

Problems from Estonian mathematical contests 1998/1999

Problems from the final round

10th grade

1. Find all pairs of integers (a, b) such that $a^2 + b = b^{1999}$.

Answer: $(0; -1)$, $(0; 0)$ and $(0; 1)$.

Solution. The given condition is equivalent to $a^2 = b(b^{1998} - 1)$. If $b \geq 2$ then b and $b^{1998} - 1$ are both positive and relatively prime, hence both perfect squares. But since b^{1998} is also a perfect square, we get a contradiction. So $b \leq 1$. The cases $b = 1$, $b = 0$ and $b = -1$ all give $a = 0$. At last note that $b \leq -2$ is impossible because it implies $a^2 < 0$.

2. Find all values of a such that absolute value of one of the roots of the equation

$$x^2 + (a - 2)x - 2a^2 + 5a - 3 = 0$$

is twice of absolute value of the other root.

Answer: $\frac{5}{4}$, $\frac{7}{5}$ and $\frac{5}{3}$.

Solution. Solving the equation, one gets

$$\begin{aligned} x &= \frac{-(a-2) \pm \sqrt{(a-2)^2 - 4 \cdot (-2a^2 + 5a - 3)}}{2} = \\ &= \frac{2-a \pm \sqrt{a^2 - 4a + 4 + 8a^2 - 20a + 12}}{2} = \\ &= \frac{2-a \pm \sqrt{9a^2 - 24a + 16}}{2} = \frac{2-a \pm \sqrt{(3a-4)^2}}{2}, \end{aligned}$$

so

$$x_1 = \frac{2-a-(3a-4)}{2} = -2a+3, \quad x_2 = \frac{2-a+(3a-4)}{2} = a-1.$$

Assume $|x_1| = 2|x_2|$, i.e. $|-2a+3| = 2 \cdot |a-1|$. The numbers $a-1$ and $-2a+3$ are negative iff $a < 1$ and $a > \frac{3}{2}$ respectively. Thus in the case $a < 1$ the condition reduces

to $-2a+3 = 2(1-a)$, and in the case $a > \frac{3}{2}$ the condition reduces to $2a-3 = 2(a-1)$.

Hence both cases are impossible. So $1 \leq a \leq \frac{3}{2}$, which gives $-2a+3 = 2(a-1)$ and

$$a = \frac{5}{4}.$$

Now assume $|x_2| = 2|x_1|$, i.e. $|a-1| = 2 \cdot |-2a+3|$. In the case $a < 1$ this condition reduces to $1-a = 2(-2a+3)$, which gives $a = \frac{5}{3}$, but this is not sound with the case

assumption $a < 1$. In the case $1 \leq a \leq \frac{3}{2}$ the condition reduces to $a-1 = 2(-2a+3)$ which gives $a = \frac{7}{5}$, and in the case $a > \frac{3}{2}$ the condition reduces to $a-1 = 2(2a-3)$ which gives $a = \frac{5}{3}$.

3. The incircle of the triangle ABC , with the center I , touches the sides AB , AC and BC in the points K , L and M respectively. Points P and Q are taken on the sides AC and BC respectively, such that $|AP| = |CL|$ and $|BQ| = |CM|$. Prove that the difference of areas of the figures $APIQB$ and $CPIQ$ is equal to the area of the quadrangle $CLIM$.

Solution. We have $S_{API} = S_{CLI}$ since $|AP| = |CL|$ and both triangles have altitude LI . Analogously we have $S_{ALI} = S_{CPI}$ since $|AL| = |CP|$. But $\triangle ALI \cong \triangle AKI$ because the sides are respectively equal. Hence

$$S_{APIK} - S_{CPI} = S_{AKI} + S_{API} - S_{CPI} = S_{ALI} + S_{CLI} - S_{ALI} = S_{CLI}.$$

Analogously we find that

$$S_{BMIK} - S_{CQI} = S_{CMI}.$$

Thus we have

$$S_{APIQB} - S_{CPIQ} = S_{APIK} + S_{BMIK} - S_{CPI} - S_{CQI} = S_{CLI} + S_{CMI} = S_{CLIM}.$$

4. 32 stones, with pairwise different weights, and lever scales without weights are given. How to determine by 35 scaling, which stone is the heaviest and which is the second by weight?

Solution. At first we determine the heaviest stone by 31 scalings and find at the same time 5 stones, one of which must be the second by weight. This can be achieved by pairing all the 32 stones and comparing the stones in each pair; after that pairing the heavier stones and comparing again the stones in each pair, etc. This takes $16 + 8 + 4 + 2 + 1 = 31$ scalings. By the last scaling we find the heaviest stone and the second by weight may be only one of these five which have been in pair with the heaviest.

To complete the solution we must determine by 4 scaling, which of these 5 stones is the heaviest. It can be done linearly.

5. Let C be an interior point of line segment AB . Equilateral triangles ADC and CEB are constructed to the same side from AB . Find all points which can be the midpoint of the segment DE .

Answer: the midline of the equilateral triangle constructed on the segment AB which is parallel to AB (endpoints excluded).

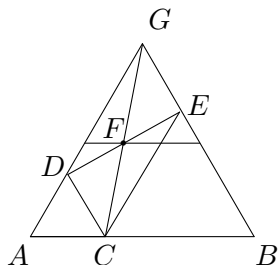


Figure 1

Solution. Let F be the midpoint of the segment DE . Lengthen the segments AD and BE till intersecting in point G (see figure 1). The location of the point G does not depend on the point P and the quadrangle $ECDG$ is a parallelogram with the point F as the midpoint of its diagonals. While the point C moves along AB from A to B , point F moves from the midpoint of AG to the midpoint of BG along the segment connecting these points. According to the condition of the problem, the endpoints must be excluded.

11th grade

1. Find all pairs of integers (m, n) such that

$$(m - n)^2 = \frac{4mn}{m + n - 1}.$$

Answer: the pairs $(k, -k)$ and $\left(\frac{k(k+1)}{2}, \frac{k(k-1)}{2}\right)$, (where k is an arbitrary integer), $(1, 0)$ and $(0, 1)$ excluded.

Solution. By multiplying both sides of the equation with the denominator of the right side, we get $(m - n)^2(m + n - 1) = 4mn$, which gives $(m + n)^2 = (m - n)^2(m + n)$. Hence $m + n = 0$ or $m + n = (m - n)^2$. The first case gives $n = -m$, i.e. all the pairs $(k, -k)$, where k is integer, are suitable. In the second case take $m - n = k$, then $m + n = k^2$ and $m = \frac{k^2 + k}{2}$, $n = \frac{k^2 - k}{2}$. To ensure the denominator of the fraction in the problem is not zero the condition $m + n \neq 1$ must be added, so $k \neq 1$ and $k \neq -1$.

2. Find the value of the expression

$$f\left(\frac{1}{2000}\right) + f\left(\frac{2}{2000}\right) + \dots + f\left(\frac{1999}{2000}\right) + f\left(\frac{2000}{2000}\right) + f\left(\frac{2000}{1999}\right) + \dots + f\left(\frac{2000}{1}\right)$$

assuming $f(x) = \frac{x^2}{1 + x^2}$.

Answer: $1999 \frac{1}{2}$.

Solution. One gets the answer directly using the fact that for any non-zero real number x ,

$$\begin{aligned} f(x) + f\left(\frac{1}{x}\right) &= \frac{x^2}{1 + x^2} + \frac{\left(\frac{1}{x}\right)^2}{1 + \left(\frac{1}{x}\right)^2} = \frac{x^2}{1 + x^2} + \frac{1}{x^2} \cdot \frac{1}{1 + \frac{1}{x^2}} = \\ &= \frac{x^2}{1 + x^2} + \frac{1}{1 + x^2} = 1. \end{aligned}$$

3. For the given triangle ABC , prove that a point X on the side AB satisfies the condition $\overrightarrow{XA} \cdot \overrightarrow{XB} + \overrightarrow{XC} \cdot \overrightarrow{XC} = \overrightarrow{CA} \cdot \overrightarrow{CB}$ iff X is the basepoint of the altitude or median of the triangle ABC ($\vec{v} \cdot \vec{u}$ denotes the scalar product of vectors \vec{v} and \vec{u}).

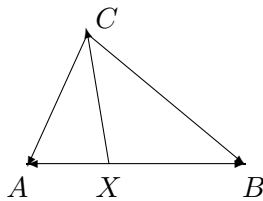
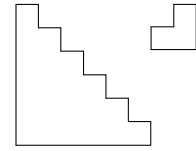


Figure 2

Solution. Since $\overrightarrow{CA} = \overrightarrow{XA} - \overrightarrow{XC}$ and $\overrightarrow{CB} = \overrightarrow{XB} - \overrightarrow{XC}$ (see figure 2), the condition in the problem is equivalent to the condition $\overrightarrow{XA} \cdot \overrightarrow{XB} + \overrightarrow{XC} \cdot \overrightarrow{XC} = (\overrightarrow{XA} - \overrightarrow{XC}) \cdot (\overrightarrow{XB} - \overrightarrow{XC})$. Transforming this, we get the equation $\overrightarrow{XC} \cdot (\overrightarrow{XA} + \overrightarrow{XB}) = 0$.

This condition holds iff $\overrightarrow{XA} + \overrightarrow{XB} = 0$ or $\overrightarrow{XC} \perp \overrightarrow{XA} + \overrightarrow{XB}$. The first case holds iff X is the midpoint of the side AB , the second case holds iff X is on the altitude.

4. For which values of n it is possible to cover the side wall of staircase of n steps (for $n = 6$ in the figure) with plates of shown shape? The width and height of each step is 1 dm, the dimensions of plate are 2×2 dm and from the corner there is cut out a piece with dimensions 1×1 dm.



Answer: $n = 3k$ or $n = 3k + 2$, except $n = 3$ and $n = 5$.

Solution. The area of the side wall of the staircase is $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ dm², the area of each plate is 3 dm². Thus one of the numbers n and $n + 1$ must be divisible by 3, i.e. it isn't possible to plate the side wall of the staircase if $n \equiv 1 \pmod{3}$.

Consider the cases $n = 2$, $n = 3$, $n = 5$. Clearly, if $n = 2$ then the plating is possible. If $n = 3$ then one of the plates must be in the vertex of the wall and it is not possible to cover the remaining part with plates with required shape (see figure 3). If $n = 5$ then two plates must be in vertices of the wall. The remaining part has shape 3×3 and thus can't be covered with 3 plates, because otherwise one of these plates must cover two different vertices of the 3×3 square.

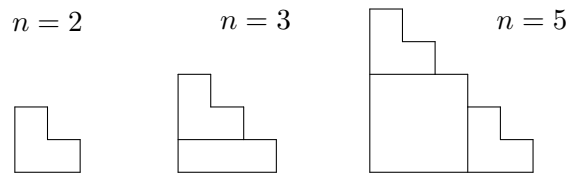


Figure 3

In the cases $n = 6$ and $n = 9$ the wall can be covered as shown in figure 4.

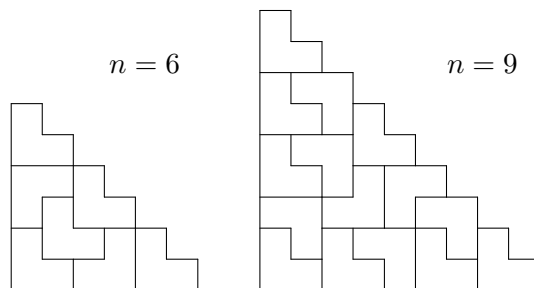


Figure 4

Now we give a method how to construct plating in case $n = 3k + 6$ from plating in case $n = 3k$. Thus the wall can be plated for any $n = 3k$, $k > 1$. For this leave unplated the lower ribbon of the wall with height 6 dm and plate the upper part as in case $n = 3k$. The ribbon can be plated as shown in figure 5.

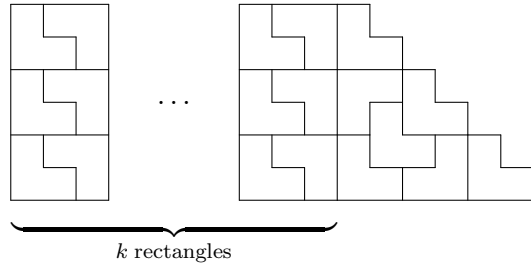


Figure 5

At last we show how to get the plating in the case $n = 3k + 2$ from the plating in the case $n = 3k$. This implies the wall can be plated for any $n = 3k + 2$, $k > 1$. For this leave unplated the lower ribbon with height 2 dm and plate the upper part as in case $n = 3k$. The ribbon can be covered as shown in figure 6.

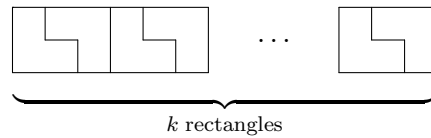


Figure 6

- On the squares a1, a2, ..., a8 of a chessboard there are respectively $2^0, 2^1, \dots, 2^7$ grains of oat, on the squares b8, b7, ..., b1 respectively $2^8, 2^9, \dots, 2^{15}$ grains of oat, on the squares c1, c2, ..., c8 respectively $2^{16}, 2^{17}, \dots, 2^{23}$ grains of oat etc. (so there are 2^{63} grains of oat on the square h1). A knight starts moving from some square and eats after each move all the grains of oat on the square to which it had jumped, but immediately after the knight leaves the square the same number of grains of oat reappear. With the last move the knight arrives to the same square from which it started moving. Prove that the number of grains of oat eaten by the knight is divisible by 3.

Solution: The arrangement of grains of oat described in the problem implies that on each white square of the chessboard there are 2^{2n} grains, while on each black square there are 2^{2n+1} grains, where $n = 0, 1, 2, \dots, 31$. Therefore the number of grains on any black square is congruent to 1 modulo 3 and the number of grains on any white square is congruent to 2 modulo 3. Since the knight moves always from a square of one colour to a square of the other colour the number of grains it eats with each two consecutive move is divisible by 3. But knight makes an even number of moves, because the initial and final square are of the same colour. Hence the assertion of the problem holds.

12th grade

- Let a, b, c and d be non-negative integers. Prove that the numbers $2^a 7^b$ and $2^c 7^d$ give the same remainder when divided by 15 iff the numbers $3^a 5^b$ and $3^c 5^d$ give the same remainder when divided by 16.

Solution. First we show that if $|a' - a| = |b' - b| = 2$ then $2^{a'} 7^{b'} \equiv 2^a 7^b \pmod{15}$ and $3^{a'} 5^{b'} \equiv 3^a 5^b \pmod{16}$. Indeed we can assume that $a' = a + 2$. If $b' = b + 2$, we obtain

$$2^{a'} 7^{b'} = 2^{a+2} 7^{b+2} = 2^a 7^b \cdot (2 \cdot 7)^2 \equiv 2^a 7^b \cdot (-1)^2 = 2^a 7^b \pmod{15}$$

and

$$3^{a'}5^{b'} = 3^{a5^b} \cdot 3^{25^2} = 3^{a5^b} \cdot (3 \cdot 5)^2 \equiv 3^{a5^b} \cdot (-1)^2 = 3^{a5^b} \pmod{16}.$$

If $b' = b - 2$, we can use the same relations noting that $7^4 \equiv 1 \pmod{15}$ and $5^4 \equiv 1 \pmod{16}$.

Now we prove that for every pair of non-negative integers (a, b) there exists a pair (a', b') such that $2^{a7^b} \equiv 2^{a'}7^{b'} \pmod{15}$, $3^{a5^b} \equiv 3^{a'}5^{b'} \pmod{16}$, $a' \in \{0, 1, 2, 3\}$ and $b' \in \{0, 1\}$. We conclude that both of the exponents can be changed by a number divisible by 4 without changing the remainder of dividing by the required number. Thus we can consider only the case where $a, b \in \{0, 1, 2, 3\}$. If $b \leq 1$, take $a' = a$ and $b' = b$; if $b > 1$ then $b' = b - 2$ and a' can be chosen from the set $\{0, 1, 2, 3\}$ so that it differs from the number a exactly by 2.

It remains to prove that the remainders of the numbers $2^{a'}7^{b'}$ when divided by 15 and the remainders of the numbers $3^{a'}5^{b'}$ when divided by 16 are pairwise different if the numbers a' and b' come from the abovementioned sets. This can be seen from the following tables.

Number	The remainder of dividing by 15	Number	The remainder of dividing by 16
2^07^0	1	3^05^0	1
2^17^0	2	3^15^0	3
2^27^0	4	3^25^0	9
2^37^0	8	3^35^0	11
2^07^1	7	3^05^1	5
2^17^1	14	3^15^1	15
2^27^1	13	3^25^1	13
2^37^1	11	3^35^1	7

2. Find the value of the integral

$$\int_{-1}^1 \ln(x + \sqrt{1+x^2}) dx.$$

Answer: 0.

Solution. Let $f(x) = \ln(x + \sqrt{1+x^2})$. We always have $1+x^2 > x^2$, so $\sqrt{1+x^2} > |x|$, thus $f(x)$ is defined for every x . We will show that the function $f(x)$ is odd, i.e. $f(-x) = -f(x)$. Indeed,

$$\begin{aligned} f(-x) &= \ln(-x + \sqrt{1+(-x)^2}) = \ln\left(\frac{(\sqrt{1+x^2}-x)(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}+x}\right) = \\ &= \ln\left(\frac{1+x^2-x^2}{\sqrt{1+x^2}+x}\right) = \ln\left(\frac{1}{x+\sqrt{1+x^2}}\right) = \\ &= -\ln(x + \sqrt{1+x^2}) = -f(x). \end{aligned}$$

Now let

$$I = \int_{-1}^1 f(x) dx, \quad I_1 = \int_{-1}^0 f(x) dx, \quad I_2 = \int_0^1 f(x) dx .$$

Making the substitution $x = -t$ in the integral I_1 , we obtain

$$I_1 = - \int_1^0 f(-t) dt = \int_0^1 f(-t) dt = - \int_0^1 f(t) dt = -I_2 .$$

Consequently

$$I = I_1 + I_2 = -I_2 + I_2 = 0 .$$

3. Prove that the line segment, joining the orthocenter and the intersection point of the medians of the acute-angled triangle ABC is parallel to the side AB iff $\tan \angle A \cdot \tan \angle B = 3$.

Solution: Let the basepoints of the altitudes drawn from the vertices A and C be D and E respectively and let the orthocenter be H (see figure 7). Note first that H lies on the line parallel to AB and passing through the intersection point of the medians if and only if $\frac{|CE|}{|EH|} = 3$, thus it is enough to prove that $\frac{|CE|}{|EH|} = \tan \angle A \cdot \tan \angle B$. As $\angle AHE = \angle CHD$, we also have $\angle EAH = \angle BCE$, i.e. the right triangles CEB and AEH are similar and $\tan \angle B = \frac{|CE|}{|EB|} = \frac{|AE|}{|EH|}$. Noting that $\tan \angle A = \frac{|CE|}{|AE|}$, we obtain the necessary relation.

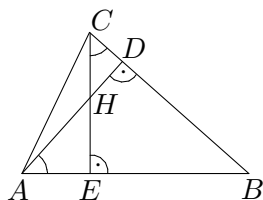


Figure 7

4. Let us put pieces on some squares of $2n \times 2n$ chessboard in such a way that on every horizontal and vertical line there is an odd number of pieces. Prove that the whole number of pieces on the black squares is even.

Solution: Enumerate all the horizontal and vertical lines by numbers $1, \dots, 2n$ and assume that the square $(1, 1)$ is black (this does not restrict generality). Let A be the number of pieces on the squares with even horizontal and vertical line number, B the number of pieces on the squares with odd horizontal and vertical line number and C the number of pieces on the squares with even horizontal and odd vertical line number. Then $A + C$ is the number of pieces with even horizontal line number and $B + C$ the number of pieces with odd vertical line number. As the number of such horizontal and vertical lines is the same, these numbers have the same parity and thus the number of pieces on the black squares $A + B = (A + C) + (B + C) - 2C$ is even.

5. The numbers $0, 1, 2, \dots, 9$ are written (in some order) on the circumference. Prove that
- there are three consecutive numbers with the sum being at least 15;
 - it is not necessarily the case that there exist three consecutive numbers with the sum more than 15.

Solution: a) The sum of numbers on the circumference is 45 and thus adding the ten triple sums gives $3 \cdot 45 = 135$. Note that in every two neighbouring triples there are two common numbers and the third one is different, so their sums must be different. Hence, if the largest sum would be 14, we should have the sums 13 and 14 altering. But then the sum of every six-tuple should be 27 which is not possible, because the sums of two neighbouring six-tuples must also be different.

b) Write on the circumference the numbers in the following order: 3, 8, 1, 5, 9, 0, 6, 7, 2, 4. Now it is elementary to check that no triple gives the sum greater than 15.

Problems from the open contests

Younger group (up to 10th grade)

1. John knows $n \geq 3$ positive real numbers and he writes them all on the blackboard (every number may occur more than once). Mary writes under every number the arithmetic mean of the other $n - 1$ numbers and then deletes the initial numbers. She repeats the process for 1998 times. After that Mary notices that there are exactly the initial numbers on the blackboard. How many different real numbers does John know?

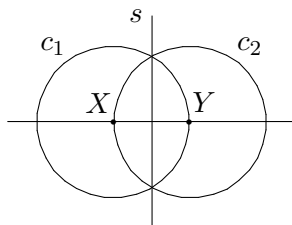
Answer: John knows one positive real number.

Solution. Let M be the greatest of the initial numbers. For arbitrary b_1, \dots, b_{n-1}

$$\frac{b_1 + \dots + b_{n-1}}{n-1} \leq \frac{\overbrace{M + \dots + M}^{n-1}}{n-1} = M$$

and the equality holds if and only if $b_1 = \dots = b_{n-1} = M$. If there were at least two numbers less than M on the blackboard then all the next numbers would also be less than M . And then of course on the 1998th time there cannot be M on the blackboard. If only one of the numbers on the blackboard is less than M then the next time there would be already two numbers less than M . Therefore all the initial numbers were equal and John knows only one positive real number.

2. Two different points X and Y are chosen in the plane. Find all the points Z in this plane for which the triangle XYZ is isosceles.



Answer: All the points on the circumferences c_1 and c_2 except the five points lying on the line XY (see the figure).

Solution. If the points X , Y and Z are the vertices of an isosceles triangle then $|XZ| = |XY|$, $|YZ| = |XY|$ or $|XZ| = |YZ|$. The points Z such that $|XZ| = |XY|$ lie on the circumference c_1 . The points Z such that $|YZ| = |XY|$ are situated on the circumference c_2 . The points Z such that $|XZ| = |YZ|$ are situated on the midperpendicular s of the segment XY . It is clear that the points determine a triangle if and only if they do not lie on the same line.

3. Prove that for every integer k the following assertions are equivalent (if one is true then the other is also true and vice versa):
- exist nonnegative integers a and b so that $k = a^2 + b^2 + ab$,
 - exist nonnegative integers c and d so that $k = c^2 + d^2 - cd$.

Solution. We shall use the relations

$$n^2 + m^2 + nm = (n + m)^2 + m^2 - (n + m)m, \quad (1)$$

$$n^2 + m^2 - nm = (n - m)^2 + m^2 + (n - m)m. \quad (2)$$

Let k be expressed in the form $k = a^2 + b^2 + ab$. Denoting $a = n$, $b = m$ on the left side and $c = n + m$, $d = m$ on the right side of the relation (1) we obtain

$$a^2 + b^2 + ab = k = c^2 + d^2 - cd.$$

Let now k be expressed in the form $k = c^2 + d^2 - cd$. Without loss of generality we can assume that $c \geq d$. Denoting $c = n$, $d = m$ on the left side and $a = n - m$, $b = m$ on the right side of the relation (2) we obtain

$$c^2 + d^2 - cd = k = a^2 + b^2 + ab.$$

4. Find all the four-digit numbers n such that multiplying n by $\frac{9}{2}$ we obtain the number which consists of the same digits as n but in the opposite order.

Answer: 1818 and 1998.

Solution. Let $n = \overline{abcd}$. Since $4,5 \cdot \overline{abcd} < 10000$ then $a = 1$ or $a = 2$. The digit d has to be even. At the same time multiplying it by $\frac{9}{2}$ the last digit of the product must be 1 or 2. Consequently $a = 2$ and $d = 6$ or $a = 1$ and $d = 8$. Let us examine these cases separately.

1) If $a = 2$ and $d = 6$ then $\frac{9}{2} \cdot \overline{abcd} > 9000$ which contradicts $d = 6$.

2) If $a = 1$ and $d = 8$ then we obtain

$$\begin{aligned} (1000 + 100b + 10c + 8) \cdot \frac{9}{2} &= 8000 + 100c + 10b + 1 \\ 3465 + 55c &= 440b \\ 63 + c &= 8b. \end{aligned}$$

This equation has two solutions: $c = 1$, $b = 8$ and $c = 9$, $b = 9$. These values will give us the numbers 1818 and 1998.

1. The teacher wrote an addition exercise on the blackboard

$$\frac{A}{B} + \frac{C}{D} + \frac{E}{F} = \text{---} ,$$

where A, B, C, D, E, F are positive integers, all three fractions are reduced and their denominators are pairwise relatively prime. The pupil adds the fractions writing the least common multiple of the denominators of the summands as the denominator of the result. Prove that the fraction that the pupil writes is reduced.

Solution. As the denominators of the fractions are pairwise relatively prime, their least common multiple is BDF and so the resulting fraction is in form $\frac{ADF + CBF + EBD}{BDF}$.

Let's suppose antithetically that this fraction is reducible by some number greater than one. Then there exists a prime number p which divides both the numerator and the denominator of this fraction. As the product BDF is divisible by the prime number p so one of terms is divisible by it: without loss of generality we can assume that this term is B . But then two summand in sum $ADF + CBF + EBD$ are divisible by p and as the sum is divisible by p by assumption, the summand ADF is divisible by p as well. Therefore either A, D or F is divisible by p . In the first case we get a contradiction with the fraction $\frac{A}{B}$ being reduced, in other cases with assumption that the denominators of the initial fractions were pairwise relatively prime.

Comment. Analogical assertion can be proved in case where there are more than three fractions as summands.

2. Prove that the value of the expression

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + n}}}$$

is not an integer for any integer n .

Solution. The expression given in the problem is not defined when $n = -1$ or $n = -2$. The value of this expression is $1 + \frac{n+2}{2n+3}$ what is an integer iff $\frac{n+2}{2n+3}$ is an integer i.e. the number $2n+3$ divides the number $n+2$. We will show that no such integer exists. As $n \neq -1$ and $n \neq -2$ then $|3+2n| \neq 1$. If there exists an integer n for which the fraction $\frac{n+2}{2n+3}$ could be reduced with some number $d > 1$ then both $2n+3$ and $n+2$ would be divisible by d , so $2 \cdot (n+2) = 2n+4$ would be divisible by d as well and $(2n+4) - (2n+3) = 1$ would be divisible by d —a contradiction. Therefore the fraction $\frac{n+2}{2n+3}$ is reduced for all integers $n \neq -1, -2$ and the value of given expression can't be an integer for any integer n .

3. On the plane there are two non-intersecting circles with equal radii and with centres O_1 and O_2 , line s going through these centres, and their common tangent t . The third circle osculates these two circles in points K and L respectively, line s in point M and line t in point P . The point of tangency of line t and the first circle is N .
- Find the length of the segment O_1O_2 .
 - Prove that the points M, K and N lie on the same line.

Answer: a) $2\sqrt{2}R$.

Solution. a) The radius of the third circle is obviously $\frac{R}{2}$; let its centre be O_3 (see figure 8) and $|O_1O_2| = c$. Considering that the triangle O_1MO_3 is a right triangle with the hypotenuse $R + \frac{R}{2} = \frac{3R}{2}$ and legs $\frac{R}{2}$ and $\frac{c}{2}$, we get $\left(\frac{R}{2}\right)^2 + \left(\frac{c}{2}\right)^2 = \left(\frac{3R}{2}\right)^2$, from where $R^2 + c^2 = (3R)^2$, $c^2 = 8R^2$ and $c = 2\sqrt{2}R$.

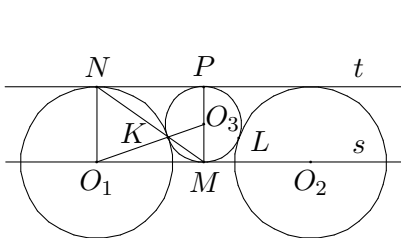


Figure 8

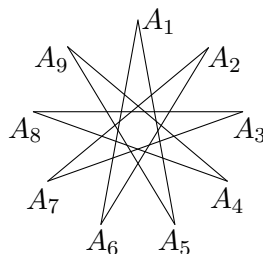


Figure 9

b) Let's denote $\angle PMK = \alpha$. From the isosceles triangle O_3MK we get $\angle MKO_3 = \alpha$ and $\angle KO_3M = \pi - 2\alpha$. As the segments PM and NO_1 are parallel the angle $\angle KO_1N = \pi - 2\alpha$ and from the isosceles triangle KO_1N we get $\angle O_1NK = \alpha$. Therefore the segments MK and NK parallel, that means the points N , K and M lie on the same line.

4. For which values of n ($n \geq 3$) is it possible to draw on a plane such a closed broken line consisting of n links that every link has exactly one point in common with every other link so that this point is an end point or an inner point for both links, and no point on the plane is an end point for more than two links?

Answer: it is possible iff n is odd.

Solution. When $n = 2k + 1$ is odd we can get the necessary construction in the following way. Let $A_1, A_2, \dots, A_{2k+1}$ be the vertices of a regular $(2k+1)$ -gon. We draw a segment from every vertex A_i to vertices A_{i+k} (or $A_{(i+k)-(2k+1)}$, if $i+k > 2k+1$) and A_{i-k} (or $A_{(i-k)+(2k+1)}$, if $i-k < 1$). We can see that a closed broken line is obtained. Indeed, from one side every vertex is connected to exactly two other vertices, from the other side it is possible to reach any vertex from any other vertex, moving by links. To be convinced in the latter assertion it is enough to notice that starting from the vertex A_i we can reach the vertex A_{i+1} using two links and hence continuing in the same way any other vertex. In addition we have to prove that every link has exactly one common point with every other link (either end point or inner point). Without loss of generality let's consider the link A_1A_{k+1} . All the other links A_iA_j ($i < j$) can be divided into three classes:

- 1) $i = 1, j = k + 2$; in this case these links have a common end point A_1 ;
- 2) $i = k, j = 2k + 1$; in this case these links have a common end point A_k ;
- 3) $1 < i < k + 1, k + 1 < j < 2k + 1$; in this case the vertices A_i and A_j lie on different sides of the line A_1A_k and because of convexity of initial regular $(2k+1)$ -gon the links A_1A_k and A_iA_j have a common inner point.

An example for the case $n = 9$ is given on the figure 9.

Now let's show that no broken line having an even number of links does not satisfy the requested conditions. Let's denote the vertices of the broken line in the order of passing by B_1, B_2, \dots, B_n . Let's consider the line B_1B_2 . As every segment $B_3B_4, B_4B_5, \dots, B_{n-1}B_n$ has to intersect it, the vertices B_3, B_5, \dots, B_{n-1} lie on one side and the vertices B_4, B_6, \dots, B_n on the other side of that line. Therefore the segments B_2B_3 and B_nB_1 can't have common points.

5. Two palmists were asked several questions about the life of mr. X, each of which had to be answered with „yes“ or „no“. The palmist A answered correctly to $\frac{22}{43}$ of questions which were answered uncorrectly by the palmist B , palmist B answered correctly to $\frac{4}{7}$ of questions which were answered uncorrectly by the palmist A . To how many questions did the palmists A and B give the same answer, when the palmist A answered correctly 51% of all questions?

Answer: 50%.

Solution. As the palmist A answered uncorrectly to 49% of all questions there were $\frac{4}{7} \cdot 49 = 28$ per cent questions that were answered uncorrectly by A but correctly by B and 21% questions that both answered uncorrectly. On the other side, as A answered correctly to $\frac{22}{43}$ of questions answered uncorrectly by B , then the part of questions answered uncorrectly by both palmists was $\frac{21}{43}$ among questions answered uncorrectly by palmist B . As both palmists answered uncorrectly 21% of all questions, the palmist B answered uncorrectly 43% of all questions and there were 22% of questions that were answered uncorrectly by B but correctly by A . So there were $28 + 22 = 50$ pro cent of questions that were answered differently by A and B and so their opinion coincided in another half of questions.

Older group (11th–12th grade)

2. Let a be an integer. Find the all real solutions of the equation

$$[x] = ax + 1$$

where $[x]$ denotes the integer part of x .

Answer: If $a < -1$ then $x = -\frac{1}{a}$; if $a = 0$ then every such real number x is suitable that $1 \leq x < 2$; if $a = 2$ then $x = -\frac{3}{2}$ or $x = -1$; if $a > 2$ then $x = -\frac{2}{a}$; if $a = -1$ or $a = 1$ then the equation has no solutions.

Solution. We will use the inequalities $x - 1 < [x] \leq x$ holding for all real numbers x .

1) If $a = 0$ then $[x] = 1$ that is $1 \leq x < 2$.

2) Since $[x] \leq x < x + 1$, the equation has no solutions if $a = 1$.

3) Let $a \geq 2$. We will obtain the inequalities $(a - 1)x > -2$ and $(a - 1)x \leq -1$ that is

$$-\frac{2}{a-1} < x \leq -\frac{1}{a-1}.$$

Since $-\frac{2}{a-1} \geq -2$ and $-\frac{1}{a-1} < 0$, we get two possibilities if $a = 2$: $[x] = -2$ or $[x] = -1$, the corresponding values of x are $x = -\frac{3}{2}$ and $x = -1$ (these are really solutions). If $a > 2$ then the only possibility is $[x] = -1$ and $x = -\frac{2}{a}$ which satisfies the equation.

4) Let $a \leq -1$. Then $a - 1 < 0$ and we obtain the inequalities

$$\frac{2}{1-a} > x \geq \frac{1}{1-a}.$$

Since $\frac{1}{1-a} > 0$ and $\frac{2}{1-a} \leq 1$ then consequently $[x] = 0$. The possibility $a = -1$ gives us a contradiction, if $a < -1$ then the solution is $x = -\frac{1}{a}$.

3. A $n \times m$ -table filled with positive integers was written on the paper. John wrote after each row the greatest common divisor of the numbers of this row and below each column the least common multiple of the numbers of this column. Let a be the least common multiple of the column of the greatest common divisors and let b be the greatest common divisor of the row of the least common multiples. Prove that b is divisible by a .

Solution. Let c_{ij} denote the element of the i th row and the j th column. We fix the row i and the column j arbitrarily. Then the least common multiplier of the numbers of the j th column is divisible by c_{ij} and c_{ij} is divisible by the greatest common divisor of the numbers of the i th row. Thus the number written below the j th column is divisible by the number written after the i th row. Since j was chosen arbitrarily, the number written after the i th row is the common factor of all the numbers written below the columns. Thus the greatest common factor b of all the numbers written below the columns is divisible by the number written after the i th row. Since i was chosen arbitrarily, the number b is the common multiple for all the numbers written after the rows. Thus b is divisible by the least common multiple a of all the numbers written after the rows.

4. On the conference of linguists there were $n \geq 3$ participants who could speak altogether 14 different languages. It is known that for every three linguists existed a language that was spoken by all three. But every language was spoken by no more than a half of linguists. Find the minimal possible value of n .

Answer: The minimal possible value of n is 8.

Solution. If $n \leq 5$ then we choose an arbitrary triple and we get a language that is spoken by at least 3 that is more than a half of the participants, contradicting the conditions of the problem. If $n = 6$ or $n = 7$ then we choose arbitrary 6v linguists. It is possible to form 20 different triples of the linguists. By pigeon-hole principle there exists a language that is the same for at least two triples and therefore is spoken by at least 4 linguists. This is again a contradiction because 4 is more than a half of 6 or 7. If $n = 8$ then it is possible to find the construction satisfying the conditions of the problem:

```

1 A C E G I K M
2 A C E H J L N
3 A D F G I L N
4 A D F H J K M
5 B C F G J K N
6 B C F H I L M
7 B D E G J L M
8 B D E H I K N

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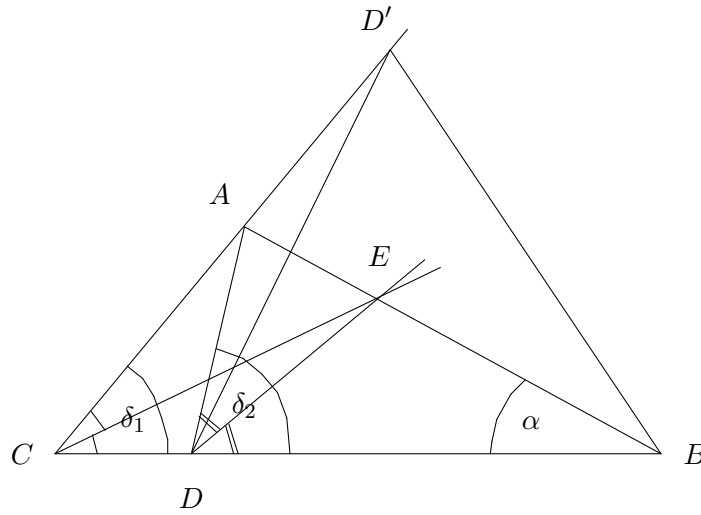
(here the letters A, \dots, N denote the languages and the numbers $1, \dots, 8$ denote the participants).

Comment: Constructing the example for the case $n = 8$ we can use the following conditions that have to be satisfied (why?):

- a) Every language is spoken by exactly 4 linguists;
- b) Every linguist can speak exactly 7 languages;
- c) For every three linguists there is one and only one language spoken by all of them.

5. On the side BC of the triangle ABC a point D different from B and C is chosen so that the bisectors of the angles ACB and ADB intersect on the side AB . Let D' be the symmetrical point to D with respect to the line AB . Prove that the points C , A and D' are on the same line.

Solution. Let E be the intersection point of the bisectors of the angles ACB and ADB and let $\angle ACB = \delta_1$, $\angle ADB = \delta_2$ and $\angle CBA = \alpha$.



By the property of the bisector for the triangles ABC and ABD we get

$$\frac{|AC|}{|BC|} = \frac{|AE|}{|BE|} = \frac{|AD|}{|BD|},$$

and by the law of sines for the same triangles

$$\frac{|AC|}{|BC|} = \frac{\sin \alpha}{\sin(180^\circ - \alpha - \delta_1)} = \frac{\sin \alpha}{\sin(\alpha + \delta_1)},$$

$$\frac{|AD|}{|BD|} = \frac{\sin \alpha}{\sin(180^\circ - \alpha - \delta_2)} = \frac{\sin \alpha}{\sin(\alpha + \delta_2)}.$$

Consequently

$$\frac{\sin \alpha}{\sin(\alpha + \delta_1)} = \frac{\sin \alpha}{\sin(\alpha + \delta_2)},$$

hence

$$\sin(\alpha + \delta_1) = \sin(\alpha + \delta_2).$$

There are two possibilities for this equation.

1) $\alpha + \delta_1 = \alpha + \delta_2$, that is $\delta_1 = \delta_2$. This is not possible because the point D is in the interior of the segment BC .

2) $\alpha + \delta_1 + \alpha + \delta_2 = \pi$, that is $\delta_1 + \delta_2 + 2\alpha = \pi$. Since the triangles ADB and $AD'B$ are congruent then $\angle AD'B = \angle ADB = \delta_2$ and $\angle ABD' = \angle ABD = \alpha$ and the point A is on the side CD' of the triangle BCD' .

1. Let a be an integer, which square divided by n gives the remainder 1. Which remainder can be obtained dividing the number a by n , if
 - a) $n = 16$;
 - b) $n = 3^k$, where k is a positive integer?

Answer: a) 1, 7, 9 or 15; b) 1 or $3^k - 1$.

Solution. a) As the square of an even number is an even number and gives an even remainder dividing by 16 the number a must be an odd number. Examining three possible cases we find that only 1, 7, 9 and 15 fit. It is possible to reduce the number of examinations needed by noticing that the squares of the numbers i and $16 - i$ give the same remainder by dividing by 16.

b) It's clear that the squares of the remainders 1 and $n - 1$ give the remainder 1 modulo n . Let's assume that the number a^2 gives the remainder 1 by dividing by $n = 3^k$ and let's prove that number a gives the remainder 1 or $3^k - 1$ by dividing by 3^k . From the assumption we get that 3^k divides the number $a^2 - 1 = (a - 1)(a + 1)$, hence there exist natural numbers i and j that $i + j = k$ where 3^i divides the number $a - 1$ and 3^j divides the number $a + 1$. If both i and j were positive then both numbers $a - 1$ and $a + 1$ would be divisible by 3, but that's impossible. Therefore one of the nest cases must occur: $i = 0$ and $j = k$, or $j = 0$ and $i = k$. Hence one of the numbers $a - 1$ and $a + 1$ is divisible by 3^k , or in another way, a gives the remainder 1 or $3^k - 1$ by dividing by 3^k .

Comment. The arguments in the part b) will still hold if number 3 is replaced by any other odd prime number.

2. A polynomial $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is called *alternating*, when $n \geq 1$ and for all $i = 1, 2, \dots, n$ the coefficients a_i and a_{i-1} are nonzero real numbers with different signs. Let $P(x)$ and $Q(x)$ be arbitrary alternating polynomials. Prove that the polynomial $R(x) = P(x)Q(x)$ is alternating.

Solution. Let's define $\alpha(x) = P(-x)$ and $\beta(x) = Q(-x)$. Then α and β are polynomials which coefficients all have same sign. Hence in the product γ of the polynomials α and β all coefficients have the same sign as well (all coefficients are positive if the signs of coefficients in α and β are same and negative if the signs of coefficients in α and β are different). Therefore

$$R(x) = P(x)Q(x) = \alpha(-x)\beta(-x) = \gamma(-x)$$

and the polynomial R is alternating.

3. Two right triangles are given, of which the incircle of the first triangle is the circumcircle of the second triangle. Let the areas of the triangles be S and S' respectively. Prove that

$$\frac{S}{S'} \geq 3 + 2\sqrt{2}.$$

Solution. Let the lengths of the legs of the outer triangle be a and b , the length of the hypotenuse be c and the radius of its incircle be r (see figure 10). Then the length of the hypotenuse of the inner triangle is $2r$ and the altitude h drawn to hypotenuse is not greater than r . Hence $S' = \frac{1}{2} \cdot 2rh = rh \leq r^2$. We know that $S = \frac{ab}{2} = \frac{a+b+c}{2}r$, from what using the inequality between arithmetical mean and geometrical mean we get

$$\begin{aligned} ab &= (a+b+c) \cdot r = ((a+b) + \sqrt{a^2+b^2}) \cdot r \geq (2\sqrt{ab} + \sqrt{2ab}) \cdot r = \\ &= (2 + \sqrt{2})\sqrt{ab} \cdot r, \end{aligned}$$

e.g. $\sqrt{ab} \geq (2 + \sqrt{2}) \cdot r$. Squaring both sides of the equation and dividing by 2 we get $S \geq (3 + 2\sqrt{2}) \cdot r^2$. Bearing in mind that $\frac{1}{S'} \geq \frac{1}{r^2}$ we see that $\frac{S}{S'} \geq 3 + 2\sqrt{2}$.

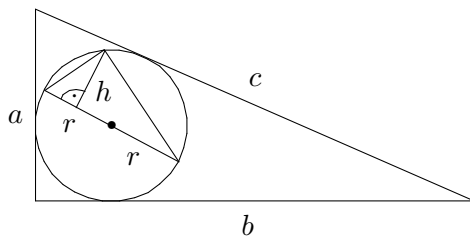


Figure 10

4. On n cells of an infinite squared board there is one piece on each cell. If one of the four neighbours of the cell containing the piece A contains the piece B and the cell behind it is empty, the piece A can be moved over the piece B to the empty cell behind it. Does there exist such a combination of pieces from which it is possible in finite number of moves to obtain the situation where the final combination of pieces is the same as in the beginning but moved by one cell in any direction, if
- $n = 1999$;
 - $n = 2000$;
 - $n = 1998$?

Answer: a) no; b) yes; c) no.

Solution. a) Let's colour the cells black and white as in the chessboard. As the number 1999 is odd, in the initial combination there are on cells of one colour (for example white) more pieces than on cells of another colour (for example black). As from one side moving the pieces as required in the problem we can move each piece only on cells coloured with same colour but from another side in the final combination we should have more pieces on black cells as on white cells we get a contradiction.

b) It's easy to see that we can move a block consisting of 2×2 pieces by one cell in any direction. We can get a suitable initial combination placing 2000 pieces on the squared board in 500 blocks as shown in the figure 11.

c) Let's colour the cells with four colours as shown in the figure 12. As the number 1998 is not divisible by four there will be on cells of some colour (for example red) more pieces than on cells of some other colour (for example blue). As we can move each piece only on cells of same colour we get a contradiction as the problem requires that we should be able to move the pieces so that in final configuration there are more pieces on blue cells than on red cells.

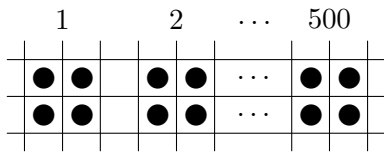


Figure 11

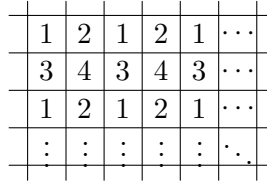


Figure 12

5. Inside the square $ABCD$ there is the square $A'B'C'D'$ so that the segments AA' , BB' , CC' and DD' do not intersect each other neither the sides of the smaller square (the sides of the larger and the smaller square do not need to be parallel). Prove that the sum of areas of the quadrangles $AA'B'B$ and $CC'D'D$ is equal to the sum of areas of the quadrangles $BB'C'C$ and $DD'A'A$.

Solution. When the centres of smaller and larger square coincide then the quadrangles $AA'B'B$, $BB'C'C$, $CC'D'D$ and $DD'A'A$ are congruent and the assertion of the problem holds. Therefore it is enough to show that the sum of areas of the quadrangles $AA'B'B$ and $CC'D'D$ does not change by any parallel displacement of smaller square (with this displacement we always can make the centres of the squares coincide).

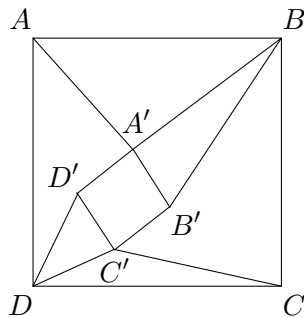


Figure 13

Without loss of generality we can assume that the vertex A' of the smaller square is not further off the side AB than the vertex B' (otherwise we can change the labels of the vertices A and B ; C and D ; A' and B' ; C' and D'). Then the diagonal $A'B$ lies inside the quadrangle $AA'B'B$ and divides it into two triangles ABA' and $A'B'B$ (see figure 13). Also the vertex C' lies not further off the side CD than the vertex D' , because of what the diagonal $C'D$ lies inside the quadrangle $CC'D'D$ and divides it into two triangles CDC' and $C'D'D$. The areas of triangles ABA' and CDC' does not change by any parallel displacement of smaller square parallel to the side AB of the larger square (because their basis AB and CD and the altitudes drawn on them don't change) and the sum of the areas of these triangles doesn't change by any parallel displacement of smaller square parallel to the side BC of the larger square (because their basis and the sum of their altitudes don't change). As every parallel displacement can be done by two displacement perpendicular to each other so the sum of the areas of the triangles ABA' and CDC' doesn't change by any parallel displacement of smaller square inside the larger square. Analogously we can see that the sum of areas of triangles $A'B'B$ and $C'D'D$ doesn't change by parallel displacement parallel to any side of smaller square (because their basis $A'B'$ and $C'D'$ and the sum of their altitudes don't change). So the sum of the areas of the quadrangles $AA'B'B$ and $CC'D'D$ does not change by any parallel displacement of the smaller square inside the larger square.