Selected problems from Estonian mathematical contests 1995-96

2nd round

1. Find all positive integers n such that $6n$ is divisible by $6+n$.

2. In a town there are three streets – T, L and H. All the citizens living in the street T tell only the truth, all the citizens from the street L only lie and in the speech of the citizens from the street H true and false sentences alternate.

   Once the fire-brigade noticed a cloud of smoke above the town and right away the phone rang.
   Caller: "There is a fire in our street!"
   Fireman: "What street is it?"
   Caller: "The street H."
   What was the street the fire-brigade had to go to?

3. Four students A, B, C and D took part in a contest of solving logical problems. They all gathered different number of points. Before the contest the students told the following.
   A: "I shall win the contest"
   B: "I am a boy and shall get the first place"
   C: "The boys are wrong, D will get the place right after me and there will be no boys after A"
   D: "C is right and there will be no girls after A"
   It is known that two contestants were boys and two girls, two of the sentences turned out to be right and two wrong. Find the final places and the genders of all the contestants.

4. An $n \times n \times n$ cube is coloured blue and cut into n^3 unit cubes. Prove that for no natural number n the number of uncoloured unit cubes and the number of unit cubes with at least one coloured face can be equal.

5. 50 different numbers are chosen from the numbers $1, 2, \ldots, 100$. Prove that from these 50 numbers we can choose two whose sum is a perfect square.

3rd round

1. Does there exist a positive integer such that its last digit is non-zero and that it becomes exactly two times bigger when the order of its digits is reversed?

2. Three children wanted to make a table-game. For that purpose they wished to enumerate the mn squares of an $m \times n$ game-board by the numbers $1, \ldots, mn$ in such way that the numbers 1 and mn lie in the corners of the board and the squares with successive numbers have a common edge. The children agreed to place the initial square (with number 1) in one of the corners but each child wanted to have the final square (with number mn) in different corner. For which numbers m and n is it possible to satisfy the wish of any of the children?

3. John and Mary play the following game. First they choose integers $n > m > 0$ and put n sweets on an empty table. Then they start to make moves alternately. A move consists of choosing a non-negative integer $k \leqslant m$ and taking k sweets away from the table (if $k = 0$, nothing happens in fact). In doing so no value for k can be chosen more than once (by none of the players) or can be greater than the number of sweets at the table at the moment of choice. The game is over when one of the players can make no more moves.

   John and Mary decided that at the beginning Mary chooses the numbers m and n and then John determines whether the performer of the last move wins or looses. Can Mary choose m and n in such way that independently of John's decision she will be able to win?

4. Prove that $1^n + 2^n + \ldots + 15^n$ is divisible by 480 for any odd $n \geqslant 5$.

5. Let p be a fixed prime number. Find all pairs (x, y) of positive integers satisfying the equation $p(x - y) = xy$.

6. In a plane there are n triangles such that any three of them have a common vertex and four of them have vertex in common. Find the greatest possible value for n.

7. In the spase there are n tetrahedra such that any two of them have two common vertices and no three of them have three vertices in common. Find the greatest possible value for n.

Final (selectional) round

1. The numbers x, y and $\dfrac{x^2 + y^2 + 6}{xy}$ are positive integers. Prove that $\dfrac{x^2 + y^2 + 6}{xy}$ is a perfect cube.

2. Let a, b, c be the sides of a triangle and α, β, γ the opposite angles of the sides respectively. Prove that if the inradius of the triangle is r then $a \sin\alpha + b\sin\beta + c\sin\gamma \geqslant 9r$.

3. Find all functions $f : R \to R$ satisfying the following conditions for all $x \in R$:

   (a) $f(x) = -f(-x)$;

   (b) $f(x+1) = f(x) + 1$;

   (c) $f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x)$, if $x \neq 0$.

4. Prove that the polynomial $P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!}$ has no zeros if n is even and has exactly one zero if n is odd.

5. Let H be the ortocenter of an obtuse triangle ABC and A_1, B_1, C_1 arbitrary points taken on the sides BC, AC, AB, respectively. Prove that the tangents drawn from the point H to the circles with diameters AA_1, BB_1, CC_1 are equal.

6. Each face of a cube is divided into $n \times n$ equal squares. The vertices of the squares are called *nodes*, so each face of the cube has $(n+1)^2$ nodes.

   (a) For $n = 2$, does there exist a closed broken line whose links are the edges of the squares which contains each node exactly once?

   (b) for n arbitrary, prove that each such broken line divides the surface area of the cube into two equal parts.

Open contests

1. Exactly one of the following statements is known to be true.

   (a) All the following statements are true;

   (b) None of the following statements is true;

   (c) At least one of the following statements is true;

   (d) All the previous statements are true;

   (e) None of the previous statements is true.

   Find the true statement.

2. A 4×4 square is divided into unit squares, some of which are then coloured and into every uncoloured square we write the number of its coloured neighbours (the squares are said to be neighbours if they have a vetrex or an edge in common). Is it possible to colour the squares in such way that in every uncoloured square there will be

   (a) the number 2;
   (b) the number 3?

3. Prove that 3^n+n^3 is divisible by 7 if and only if $3^n\cdot n^3+1$ is divisible by 7.

4. Is it possible to draw 19 lines in a plane in such way that they have exactly 95 points of intersection and no three lines have a point in common?

5. In a company mn soldiers were drawn up into m rows and n columns. First they got a command to rearrange themselves inside the rows into increasing order of tallness and after that they got a second command to perform similar rearrangement inside the columns. Prove that after the final rearrangement the soldiers in the rows still stand in increasing order of tallness.

6. In a pentagon (not necessarily convex) all the sides are of length 1 and the product of cosines of any four angles is zero. Find all possible values for the area of the pentagon.

7. The game-board has a form of rectangle with width 2 and height n. On each but one square of the board one of the $2n-1$ bricks is placed, one square is left empty. The bricks have numbers $1,2,\ldots,2n-1$ and the bricks number 1 and 2 are in the top row. A move consists of moving a brick from a square having a common edge with the empty square onto the empty square. Is it possible to interchange the places of the bricks number 1 and 2 using the finite number of described moves, if

   (a) n=2;
   (b) n=3?

8. Which of the numbers $2^{1996!}$ and $2^{1996}!$ is greater?

9. There are n petrol stations on a circular motorway. Each of the stations has a certain amount of fuel, while all the stations together have exactly the amount of fuel which is necessary to run through the motorway. Prove that it is always possible to choose the first station such that starting from that station with an empty cistern a car can run through the motorway and return to the initial point.