Estonian Math Competitions 2016/2017

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Estonian Mathematical Olympiad

http://www.math.olympiaadid.ut.ee/

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in September and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: Juniors and Seniors. In the first category, students up to the 10th grade can participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May; in recent years experimentally in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Besides the above-mentioned contests and the quiz "Kangaroo" some other regional and international competitions and matches between schools are held.

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This booklet presents the problems of the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem to be interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

O1. *(Juniors.)* Juku conjectured the following in his mathematics circle: whenever the product of two coprime integers x and y is divisible by the product of some two coprime integers a and b, at least one of x and y is divisible by a or b. Does his proposition hold?

Answer: No.

Solution. Let $x = 20, y = 21, a = 14, b = 15$. Then x and y are coprime, as they are consecutive, similarly a and b are coprime. The product $xy = 420$ is divisible by $ab = 210$ but neither of 20 and 21 is divisible by 14 or 15.

O2. *(Juniors.)* Solve the system $a^3 + b = 4c$, $a + b^3 = c$, $ab = -1$.

Answer: $a = 1$, $b = -1$, $c = 0$; $a = -1$, $b = 1$, $c = 0$; $a = 2$, $b = -\frac{1}{2}$, $c = \frac{15}{8}$; $a = -2$, $b = \frac{1}{2}$, $c = -\frac{15}{8}$.

Solution 1. From the third equation we get $b = -\frac{1}{a}$. By substituting this in the first and second equations we obtain a new system: $a^3 - \frac{1}{a} = 4c$, $a - \frac{1}{a^3} = c$. If $c = 0$, we have $a^3 = \frac{1}{a}$ and $a^4 = 1$, whence $a = 1$ or $a = -1$, since $a = 0$ is not possible. We have respectively $b = -1$ and $b = 1$. If $c \neq 0$, then by dividing in this system the sides of the first equation by the respective sides of the second equation, we obtain $(a^3 - \frac{1}{a}) : (a - \frac{1}{a^3}) = 4$. As $a^3 - \frac{1}{a} = a^2 \cdot (a - \frac{1}{a^3})$, this is equivalent to $a^2 = 4$, whence $a = \pm 2$. If $a = 2$, then $b = -\frac{1}{2}$ and $c = \frac{15}{8}$. The case $a = -2$ gives the same solution with opposite signs.

Solution 2. By adding the first and second equation we get $a(a^2 + 1) + b(b^2 + 1) = 5c$. Since $1 = -ab$ by the third equation, this equation is equivalent to $a(a^2 - ab) + b(b^2 - ab) = 5c$ or, equivalently, $(a^2 - b^2)(a - b) = 5c$. By subtracting the second equation from the first equation in the initial system we obtain $a(a^2 - 1) - b(b^2 - 1) = 3c$ and by similarly substituting from the third equation we obtain $a(a^2 + ab) - b(b^2 + ab) = 3c$ or, equivalently, $(a^2 - b^2)(a + b) = 3c$. Hence if $c = 0$, then $a - b = 0$ or $a + b = 0$. By the third equation $a = b$ is not possible. The case $a = -b$ gives two possibilities $a = 1$, $b = -1$ and $a = -1$, $b = 1$. If $c \neq 0$, we obtain $\frac{a+b}{a-b} = \frac{3}{5}$, whence $a = -4b$. By substituting into the third equation of the initial system we obtain $-4b^2 = -1$, whence $b = \pm\frac{1}{2}$. If $b = \frac{1}{2}$, then $a = -2$ and $c = -\frac{15}{8}$. The case $b = -\frac{1}{2}$ gives the same solution with opposite signs.

O3. *(Juniors.)* Find all possibilities: how many acute angles can there be in a convex polygon?

Answer: 0, 1, 2, 3.

Solution 1. A square has 0 acute angles, a right-angled trapezium has 1 acute angle, an obtuse triangle has 2 acute angles, an acute triangle has 3.

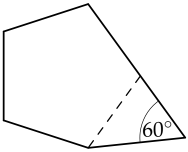


Fig. 1

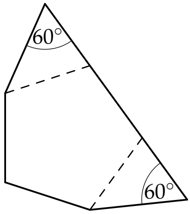


Fig. 2

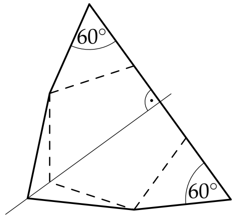


Fig. 3

Let us show that 4 or more acute angles is not possible. By moving along the boundary of the convex polygon, we turn at vertices in only one direction (for example left), and on arrival at the initial vertex we have turned a total of 360°. In an acute-angled vertex the direction changes by more than 90° (the change of direction equals the size of the respective exterior angle which is obtuse in this case). Thus if in the polygon there were 4 or more acute angles, the total turn in only those would be more than 360°.

Solution 2. In a regular pentagon all angles have size $\frac{3\cdot 180^\circ}{5}$ or 108°, thus there are 0 acute angles. By prolonging two sides in one direction, the angle of the pentagon at the moving vertex is reduced. As the angle between the extensions of two non-neighbouring sides is $180^\circ - 2\cdot(180^\circ - 108^\circ)$, which equals 36°, the receding angle can be given size 60° without losing the convexity of the polygon (Fig. 1). By prolonging the already prolonged side in the other direction, we can similarly create another acute angle of size 60° (Fig. 2). Finally let as move the opposite vertex of the prolonged side away from that side. This can be done without losing convexity until there is an acute angle also at that vertex (Fig. 3). Indeed, as in this process the unchanging angles have size 60°, totalling 120°, the convexity of the pentagon is lost only when the size of the reducing angle reaches 60°, when the pentagon becomes a regular triangle.

On the other hand, when x angles of a convex n-gon are acute, then the sum of their sizes is less than $x\cdot 90^\circ$ and the sum of the sizes of the other angles is less than $(n-x)\cdot 180^\circ$. But the sum of the sizes of the interior angles of any n-gon is $(n-2)\cdot 180^\circ$. Therefore we get $(n-2)\cdot 180^\circ < x\cdot 90^\circ + (n-x)\cdot 180^\circ$, whence $x\cdot 90^\circ < 2\cdot 180^\circ$ and $x < 4$. Thus there can only be 0 to 3 acute angles in a convex n-gon.

O4. *(Juniors.)* Does there exist a positive integer n which has exactly 9 positive divisors and whose all divisors can be placed in a 3-by-3 table such that the products of the 3 numbers in each row, each column and on each diagonal are all the same?

Answer: Yes.

Solution 1. The number 36 has 9 positive divisors 1, 2, 3, 4, 6, 9, 12, 18, 36. Let the first row be 18, 1, 12, second row 4, 6, 9, and third row 3, 36, 2. Then the product of each row, column and diagonal is 216.

Solution 2. For each prime p the number p^8 has exactly 9 divisors p^0, $p^1, \ldots, p^8$. It is known that the numbers 1 to 9 can be placed as a 3×3 magic square, with an equal sum of the numbers in each row, column and diagonal. By subtracting 1 from each number, we reduce the sum of each row, column and diagonal by 3. By replacing in the magic square each number i by the respective power p^i we obtain a placement of the divisors of p^8 in which each row, column and diagonal has an equal product.

Remark. There are other suitable examples. It can be shown that for each positive integer n with exactly 9 positive divisors its divisors can be placed as a 3×3 table, with the products of the numbers of each row, column and diagonal all equal. Consider the formula for the number of divisors

$$\delta(p_1^{\alpha_1} \ldots p_k^{\alpha_k}) = (1+\alpha_1)\ldots(1+\alpha_k).$$

To have 9 divisors we have two possibilities: a) $k = 1$ and $1 + \alpha_1 = 9$, whence $n = p^8$ for some prime p, and the divisors are $p^0, p^1, \ldots, p^8$; b) $k = 2$ and $1+\alpha_1 = 1+\alpha_2 = 3$, whence $n = p^2q^2$ for some distinct primes p and q, and the divisors are $1, p, p^2, q, pq, p^2q, q^2, pq^2, p^2q^2$. Placing the divisors as

$$\begin{array}{ccc} p^3 & p^8 & p^1 \\ p^2 & p^4 & p^6 \\ p^7 & p^0 & p^5 \end{array} \qquad\qquad \begin{array}{ccc} pq^2 & 1 & p^2q \\ p^2 & pq & q^2 \\ q & p^2q^2 & p \end{array}$$

gives respectively the product p^{12} and p^3q^3 in all directions.

O5. *(Juniors.)* Juku thought of a 3-digit number that, when reversing the order of the digits, stays the same 3-digit number. Juku noticed that when adding 2016 to that number, the 4-digit number that arises is again the same 4-digit number when reading the digits from right to left. What number did Juku think of?

Answer: 646.

Solution. Let the number be $\overline{aba}$ and let the number we get by adding 2016 be $\overline{cddc}$. Clearly c can only be 2 or 3.

If $c = 2$ then by the ones digit the only possibility is $a = 6$, and we have a carry from the ones to the tens digit. By the tens digit then $b+1+1 = d$ or $b+1+1 = d+10$. The second option is impossible, since by the hundreds digit we can only have $d = 6$, if there is no carry from the tens to the hundreds digit, and $d = 7$, if there is a carry from the tens to the hundreds digit. Thus $b + 2 = d$. Then there is no carry to the hundreds digit, hence $d = 6$ and $b = 4$.

If $c = 3$ then in adding the hundreds digits we must have a carry to the thousands digit which is possible only when $a = 9$. But by the ones digit we should have $c = 5$. The contradiction shows that this case is not possible.

O6. *(Juniors.)* a) Let a and b be arbitrary positive integers of equal parity. Can we always find noninteger numbers x and y such that $x + y$ and $ax + by$ are integers? b) The same question when a and b have different parities.

Answer: a) Yes; b) No.

Solution 1. a) By taking $x = y = \frac{1}{2}$ we have that $x + y = 1$ is an integer and so is $ax + by = \frac{1}{2}(a + b)$, since $a + b$ is even by the assumption.

b) We notice that $ax + by = a(x + y) + (b - a)y$. Assume that $x + y$ and $ax + by$ are integers. Since $a(x + y)$ is an integer, since it is a product of two integers, $(b - a)y$ must be an integer as well. But in the case $b - a = 1$ this is not possible, since y is noninteger by the assumption.

Solution 2. a) If $a = b$ then any noninteger numbers x and y whose sum is an integer will be suitable as in this case $ax + ay$ is an integer, as it is a product of two integers a and $x + y$. In the case $a \neq b$ we can take $x = \frac{1}{a-b}$ and $y = \frac{a-b-1}{a-b}$, as in that case $x + y = 1$ and

$$ax + by = \frac{a}{a-b} + \frac{b(a-b-1)}{a-b} = \frac{(b+1)(a-b)}{a-b} = b + 1.$$

b) Assume $x + y = n$ and $ax + by = m$, where n and m are integers. By interpreting this as a system of equations and solving for x and y we obtain $x = \frac{m-bn}{a-b}$ and $y = \frac{an-m}{a-b}$ (as a and b have different parities we have $a - b \neq 0$). If $a - b = 1$ then these solutions are integers. Thus we can not guarantee the existence of noninteger numbers with desired properties.

O7. *(Juniors.)* Let A and B be such points of the circle with centre O that the triangle AOB is right-angled. The perpendicular bisector of the segment AO intersects the shorter arc AB in point K. The lines KO and AB intersect in point L. Prove that the triangle KBL is isosceles.

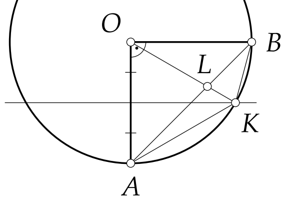


Fig. 4

Solution. Since K lies on the perpendicular bisector of the segment AO (Fig. 4) we have $KA = KO$. On the other hand $KO = AO$ since K and A are points on the circle. Hence AKO is an equilateral triangle from which we obtain that $\angle AOK = 60°$. Hence $\angle KOB = \angle AOB - \angle AOK = 90° - 60° = 30°$. As also B lies on the same circle we have $KO = BO$, hence $\angle BKO = \frac{180° - \angle KOB}{2} = \frac{180° - 30°}{2} = 75°$. Finally from the equality $AO = BO$ we obtain $\angle ABO = \frac{180° - \angle AOB}{2} = \frac{90°}{2} = 45°$. Hence $\angle BLK = \angle LOB + \angle LBO = 30° + 45° = 75°$. The triangle KBL is isosceles as it has two equal angles.

O8. *(Juniors.)* There is a finite number of lamps in an electrical scheme. Some pairs of lamps are directly connected by a wire. Every lamp is lit either red or blue. With one switch all lamps that have a direct connection with a lamp of the other colour change their colour (from red to blue or vice versa). Prove that after some number of switches all lamps have the same colour as two switches before that.

Solution. If some connected lamps are lit in different colours they both change colour upon switching, hence they are also lit differently after the switch. The same holds on each following switch. Hence no pair of connected lamps lit in different colours can disappear but more of such pairs can appear. As there are only finitely many lamps, the number of connected pairs of differently lit lamps can not grow infinitely. Hence this number stops changing after some number of switches. This means at that point a lamp either changes colour on each switch or never changes colour. Hence the colours of all lamps are the same after two consecutive switches.

O9. *(Seniors.)* Define $a_1 = 1$, and for each $n > 1$ let $a_n = n \cdot a_{\lfloor \frac{n}{2} \rfloor}$. Prove that for each $n \geqslant 12$ we have $a_n > n^2$.

Solution 1. As $a_n = n \cdot a_{\lfloor \frac{n}{2} \rfloor}$, it suffices to show that for each $n \geqslant 12$ we have $a_{\lfloor \frac{n}{2} \rfloor} \geqslant n$. By the inequalities $n \leqslant 2\lfloor \frac{n}{2} \rfloor + 1 < 3\lfloor \frac{n}{2} \rfloor$ this reduces to proving that $a_m \geqslant 3m$ for each $m \geqslant 6$. By $a_m = m \cdot a_{\lfloor \frac{m}{2} \rfloor}$ the latter reduces to proving $a_l \geqslant 3$ for $l \geqslant 3$. This is true, since $a_l = l \cdot a_{\lfloor \frac{l}{2} \rfloor} \geqslant l$.

Solution 2. By brute-force computing we obtain

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	1	2	3	8	10	18	21	64	72	100	110	216	234	294	315
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Hence for $12 \leqslant n < 16$ the claim is true.

For bigger numbers note that for each $k \geqslant 4$ we have

$$a_{2^k} = 2^k \cdot 2^{k-1} \cdot \ldots \cdot 2 \cdot 1 = 2^{k+(k-1)+\ldots+1} = 2^{\frac{k(k+1)}{2}} \geqslant 2^{2(k+1)}.$$

By simple induction we conclude that a_n increases as n increases. For arbitrary $n \geqslant 16$ pick $k \geqslant 4$ such that $2^k \leqslant n < 2^{k+1}$, giving

$$a_n \geqslant a_{2^k} \geqslant 2^{2(k+1)} = (2^{k+1})^2 > n^2.$$

O10. *(Seniors.)* Kati and Peeter play the following game. First, Kati writes a positive integer $a > 2016$ on the blackboard. Then Peeter starts to write more numbers on the blackboard, adding at each step the number $2016b + 1$ where b is the biggest number on the blackboard. Peeter wins if at some point he writes a number divisible by 2017. Otherwise Kati wins. Can Kati win, and if yes, what is the smallest number a she can write to win?

Answer: Yes, 2019.

Solution. The number $2016b + 1$ gives the same remainder upon division by 2017 as $-b + 1$. Hence the remainders upon division by 2017 are as in the sequence b, $-b + 1$, $-(-b + 1) + 1$, ... Since $-(-b + 1) + 1 = b$, this sequence has period 2, whence there are at most 2 different remainders. Hence the number a gives the win to Kati if and only if neither a nor $-a + 1$ is divisible by 2017. Thus $a = 2017$ does not give a win, as it is divisible by 2017, similarly for $a = 2018$, as $-2018 + 1$ is divisible by 2017. But for $a = 2019$ none of 2019 or $-2019 + 1$ is divisible by 2017, giving a win.

O11. *(Seniors.)* Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that for all real numbers x and y satisfy $f(x+y)f(xy) = f(x^2 - y^2 + 1)$.

Answer: $f(x) = 0$, $f(x) = 1$.

Solution. By substituting $y = 0$ into the identity we obtain for each x that $f(x)f(0) = f(1 + x^2)$. By substituting $x = 0$ into the initial identity we obtain for each y that $f(y)f(0) = f(1 - y^2)$.

If $f(0) = 0$ then from the first equality we have $f(t) = 0$ for each real number $t \geqslant 1$ (since for each $t \geqslant 1$ there exists x such that $t = 1 + x^2$) and from the second equality we have $f(t) = 0$ for each real number $t \leqslant 1$ (since for each $t \leqslant 1$ there exists y such that $t = 1 - y^2$). As a conclusion, $f(t) = 0$ for each real number t. This function also satisfies the initial identity.

If $f(0) \neq 0$ then $f(x) = \frac{f(1+x^2)}{f(0)} = f(-x)$ for each real number x, hence f is an even function. By taking $y = -x$ in the initial identity we obtain $f(0)f(-x^2) = f(1)$, implying $f(-x^2) = \frac{f(1)}{f(0)}$. As each non-positive number can be expressed as $-x^2$ and $f(x^2) = f(-x^2)$, we have $f(t) = \frac{f(1)}{f(0)}$ for each real number t. By taking $t = 1$ we obtain $f(0) = 1$, whence $f(t) = 1$ for each real number t.This function also satisfies the initial identity.

O12. *(Seniors.)* On the sides BC, CA and AB of triangle ABC, respectively, points D, E and F are chosen. Prove that

$$\frac{1}{2}(BC + CA + AB) < AD + BE + CF < \frac{3}{2}(BC + CA + AB).$$

Solution 1. Without loss of generality let $BC \geqslant CA \geqslant AB$.

Let us first prove the second inequality. Consider the circle with centre A that passes through the vertex C and consider the extension of the side CB past the vertex B up to this circle (Fig. 5). As a chord lies inside the circle, we have $AD \leqslant AC$. We also have $BE \leqslant \max(BC, BA) = BC$ and $CF \leqslant \max(CA, CB) = CB$. Since by the triangle inequality $\frac{1}{2}BC < \frac{1}{2}CA + \frac{1}{2}AB$, we obtain $AD + BE + CF \leqslant 2BC + CA < \frac{3}{2}BC + \frac{3}{2}CA + \frac{1}{2}AB < \frac{3}{2}(BC + CA + AB)$.

Let us now prove the first inequality. If the triangle is not acute then $BE \geqslant BA$ and $CF \geqslant CA$ (Fig. 6). Since by the triangle inequality we have $\frac{1}{2}BC < \frac{1}{2}CA + \frac{1}{2}AB$, we obtain $AD + BE + CF \geqslant AD + BA + CA > BA +$

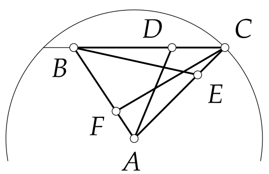


Fig. 5

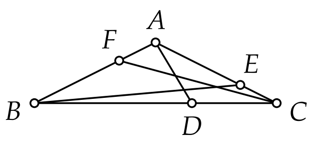


Fig. 6

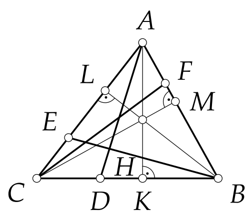


Fig. 7

$CA > \frac{1}{2}(BC + CA + AB)$. For an acute triangle we have $AD \geqslant AK$, $BE \geqslant BL$ and $CF \geqslant CM$, where AK, BL and CM are the altitudes of the triangle ABC. Let H be the intersection point of the altitudes of the triangle ABC (Fig. 7). We obtain $AD + BE + CF \geqslant AK + BL + CM > AH + BH + CH = \frac{1}{2}(BH + CH) + \frac{1}{2}(CH + AH) + \frac{1}{2}(AH + BH) > \frac{1}{2}(BC + CA + AB)$, where the last inequality follows from the triangle inequality.

Solution 2. By the triangle inequality, $AB \leqslant BD + AD$ and $AB \leqslant AE + BE$, analogous inequalities hold for sides BC and CA. As not all equality cases can hold simultaneously, adding these inequalities gives a strict inequality $2(BC + CA + AB) < BC + CA + AB + 2(AD + BE + CF)$. Collecting similar terms and dividing by 2 gives the first required inequality.

Similarly by the triangle inequality, $AD \leqslant AB + BD$ and $AD \leqslant CA + CD$; analogously for BE and CF. Again, not all equality cases can hold simultaneously, whence adding these inequalities gives a strict inequality $2(AD + BE + CF) < 2(BC + CA + AB) + BC + CA + AB$. Collecting similar terms and dividing by 2 leads to the second required inequality.

O13. *(Seniors.)* Find all positive integers n for which all positive divisors of n, taken without repetitions, can be placed into a rectangular table in such a way that each cell contains exactly one divisor, all row sums are equal and all column sums are equal.

Answer: 1.

Solution 1. Suppose that all positive divisors of n can be arranged as a rectangular table of size $k \times l$. Assume w.l.o.g. that $k \leqslant l$ (k is the number of rows). Let the sum of the numbers in each column be s; as n occurs somewhere in the table, we must have $s \geqslant n$, whereby equality can hold only if $k = 1$. For every $j = 1, 2, \ldots, l$, let d_j be the largest number in the jth column; w.l.o.g., $d_1 > d_2 > \ldots > d_l$. As the divisors of n are among $n, \frac{n}{2}, \frac{n}{3}, \ldots$, this chain of inequalities implies $d_l \leqslant \frac{n}{l}$. Since the average number in any column cannot exceed the maximum value of that column, we also have $d_l \geqslant \frac{s}{k} \geqslant \frac{n}{k}$. These inequalities together imply $\frac{n}{k} \leqslant d_l \leqslant \frac{n}{l}$. Hence $k \geqslant l$. As we assumed $k \leqslant l$, we conclude that $k = l$. Therefore all these inequalities must actually be equalities. In particular $s = n$, implying $k = l = 1$. Consequently, n has only one divisor, i.e., $n = 1$.

Solution 2. Obviously, $n = 1$ obviously meets the conditions. Suppose that we have a required arrangement of all divisors of n for some $n > 1$.

If n is a power of 2 then all divisors except 1 are even. As all numbers greater than 1 have more than one divisor, there must be at least 2 columns (w.l.o.g.) in the table. However, one column sum is odd while all other column sums are even, which contradicts the conditions of the problem.

Assume now that n has at least one odd prime divisor; let $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$, where $s \geqslant 1$, $p_1 < \ldots < p_s$ are primes and $\alpha_1, \ldots, \alpha_s$ are positive integers. Denote the number of all positive divisors of n and the sum of all positive divisors of n by $\delta(n)$ and $\sigma(n)$, respectively. As n occurs in the table, the row sums and column sums must be at least n. Since there are at least $\sqrt{\delta(n)}$ rows or columns, we must have $\sigma(n) \geqslant n\sqrt{\delta(n)}$. Denoting $f(n) = \frac{\sigma(n)}{n\sqrt{\delta(n)}}$, we can write $f(n) \geqslant 1$. The definition implies that f is (weakly) multiplicative, i.e., $f(n) = f(p_1^{\alpha_1}) \dots f(p_s^{\alpha_s})$. Furthermore, note the following two monotonicity properties:

(*) For any primes p, q, if $p < q$ then $f(p) > f(q)$;

(**) For arbitrary prime number p and positive integer k, $f(p^k) > f(p^{k+1})$.

For proving (*), we can write

$$\frac{f(q)}{f(p)} = \frac{1+q}{q\sqrt{2}} \cdot \frac{p\sqrt{2}}{1+p} = \frac{1+\frac{1}{q}}{1+\frac{1}{p}} < \frac{1+\frac{1}{p}}{1+\frac{1}{p}} = 1.$$

For proving (**), we analogously get

$$\frac{f(p^{k+1})}{f(p^k)} = \frac{1+p+\ldots+p^{k+1}}{p^{k+1}\sqrt{k+2}} \cdot \frac{p^k\sqrt{k+1}}{1+p+\ldots+p^k} = \frac{1+\frac{1}{p(1+p+\ldots+p^k)}}{\sqrt{1+\frac{1}{k+1}}}$$

$$\leqslant \frac{1+\frac{1}{p+p^{k+1}}}{\sqrt{1+\frac{1}{k+1}}} \leqslant \frac{1+\frac{1}{2+2^{k+1}}}{\sqrt{1+\frac{1}{k+1}}} = \sqrt{\frac{\left(1+\frac{1}{2+2^{k+1}}\right)^2}{1+\frac{1}{k+1}}} = \sqrt{\frac{1+\frac{1}{1+2^k}+\frac{1}{(2+2^{k+1})^2}}{1+\frac{1}{k+1}}}.$$

As $(2+2^{k+1})^2 = 4+2^{k+3}+2^{2k+2} > 2^k+2^{2k} = 2^k(1+2^k)$, we have

$$1+\frac{1}{1+2^k}+\frac{1}{(2+2^{k+1})^2} < 1+\frac{1}{1+2^k}+\frac{1}{2^k(1+2^k)} = 1+\frac{1}{2^k}.$$

Consequently,

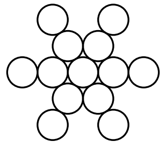
$$\frac{f(p^{k+1})}{f(p^k)} < \sqrt{\frac{1+\frac{1}{2^k}}{1+\frac{1}{k+1}}} \leqslant \sqrt{\frac{1+\frac{1}{k+1}}{1+\frac{1}{k+1}}} = 1,$$

because $2^k \geqslant k+1$ for every $k \geqslant 1$.

Note that $f(2) \cdot f(3) = \frac{3}{2\sqrt{2}} \cdot \frac{4}{3\sqrt{2}} = 1$. Thus, by (*), $f(2) > 1 > f(3)$. Now, by (**), $f(n) = f(p_1^{\alpha_1}) \cdot \ldots \cdot f(p_s^{\alpha_s}) \leqslant f(p_1) \cdot \ldots \cdot f(p_s)$. W.l.o.g., assume that $p_1 = 2$ (and $s \geqslant 2$), since otherwise, we could insert $f(2)$ into the product. Hence, by (*) and $f(3) < 1$, $f(n) \leqslant f(2) \cdot f(p_2) \cdot \ldots \cdot f(p_s) \leqslant f(2) \cdot f(3)^{s-1} \leqslant f(2) \cdot f(3) = 1$. Consequently, the required arrangement is possible only if all inequalities used in proving $f(n) \leqslant 1$ are equalities, i.e., $\alpha_1 = \ldots = \alpha_s = 1$, $p_1 = 2$, $p_2 = 3$ and $s = 2$. Thus $n = 6$. Number 6 has 4 positive divisors. So the smaller dimension of the table is at most 2; but using only two proper divisors of 6 it is impossible to obtain 6 as the sum. Hence a required arrangement of numbers $n > 1$ is impossible.

Remark. This problem, proposed by Estonia, appeared in the IMO 2016 shortlist as N2.

O14. *(Seniors.)* All numbers 1 through 13 are written in the circles of the snowflake in such a way that the sum of the five numbers on each line and the sum of the middle seven numbers are all equal. Find this sum if it is known that it is the smallest possible.



Answer: 31.

Solution. Let the sum of each ray be s and let the sum in the middle circle be a. Then

$$3s = (1 + 2 + \ldots + 13) + 2a = 91 + 2a,$$

whence $s = \frac{91+2a}{3} \geqslant \frac{93}{3} = 31$. Figure 8 shows that $s = 31$ is possible.

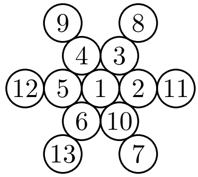


Fig. 8

O15. *(Seniors.)* Masha has an electric carousel in her garden that she rides every day. As she likes order, she always leaves the carousel in the same position after each ride. But every night three bears sneak into the garden and start turning the carousel. Bear dad turns the carousel each time by $\frac{1}{7}$ of the full circle. Bear mum turns the carousel each time by $\frac{1}{9}$ of the full circle. Bear cub turns the carousel each time by $\frac{1}{32}$ of the full circle. Every bear can turn the carousel as many times as he or she wants. In how many different positions may Masha find the carousel in the morning?

Answer: 2016.

Solution 1. As $7 \cdot 9 \cdot 32 = 2016$, all turns are integral multiples of $\frac{1}{2016}$ of the full turn. Thus the carousel can be in at most 2016 distinct positions. It remains to show that all these positions are possible. For that, we show that the bears can turn the carousel by exactly $\frac{1}{2016}$ of the full turn. Then the same sequence of operations can be repeated to obtain also $\frac{2}{2016}, \frac{3}{2016}, \ldots, \frac{2016}{2016}$ of the full turn. Exactly $\frac{1}{2016}$ of the full turn is obtained, for instance, if bear dad turns the carousel once in one direction and both bear mum and bear cub turn the carousel once in the opposite direction since $\frac{1}{7} - \frac{1}{9} - \frac{1}{32} = \frac{288-224-63}{2016} = \frac{1}{2016}$.

Solution 2. The carousel turns by an integral multiple of $\frac{1}{7}$ of the full turn due to bear dad, an integral multiple of $\frac{1}{9}$ of the full turn due to bear mum and an integral multiple of $\frac{1}{32}$ of the full turn due to bear cub. As the result, the carousel turns by $\frac{x}{7} + \frac{y}{9} + \frac{z}{32}$ of the full turn where x, y, z are some integers. As $\frac{x}{7} + \frac{y}{9} + \frac{z}{32} = \frac{288x+224y+63z}{2016}$, the carousel can be turned only by integral multiples of $\frac{1}{2016}$ of the full turn. As $\gcd(288, 224, 63) = \gcd(9 \cdot 32, 7 \cdot 32, 7 \cdot 9) = 1$, there exist integral coefficients a, b and c such that $a \cdot 288 + b \cdot 224 + c \cdot 63 = 1$. Hence taking $x = na$, $y = nb$ and $z = nc$ for any integer n, the carousel turns by exactly $\frac{n}{2016}$ of the full turn. Hence all integral multiples of $\frac{1}{2016}$ of the full turn are possible, i.e., the carousel can be left in 2016 distinct positions.

O16. *(Seniors.)* Solve the system of equation $3x + 7y + 14z = 252$, $xyz - u^2 = 2016$ for non-negative real numbers.

Answer: $x = 28, y = 12, z = 6, u = 0$.

Solution 1. From the conditions of the problem we obtain

$$252 = 3x + 7y + 14z \geqslant 3\sqrt[3]{3x \cdot 7y \cdot 14z} = 3\sqrt[3]{3 \cdot 7 \cdot 14 \cdot (2016 + u^2)}$$
$$\geqslant 3\sqrt[3]{3 \cdot 7 \cdot 14 \cdot 2016} = 3\sqrt[3]{2^6 \cdot 3^3 \cdot 7^3} = 2^2 \cdot 3^2 \cdot 7 = 252.$$

To not get a contradiction, we must have equality in both inequalities, hence $3x = 7y = 14z$ and $u = 0$. From the first equation of the system we finally obtain $3x = 7y = 14z = \frac{252}{3} = 84$, hence $x = 28$, $y = 12$ and $z = 6$.

Solution 2. The second equation implies $xyz \geqslant 2016$. Substituting the value of x from the first equation here gives $(252 - 7y - 14z)yz \geqslant 6048$, which is equivalent to $(36 - y - 2z)yz \geqslant 864$. We find the maximum of function $f(y,z) = (36 - y - 2z)yz$. Fixing $z > 0$ arbitrarily, we obtain $\frac{\mathrm{d}f(y,z)}{\mathrm{d}y} = 36z - 2yz - 2z^2 = 2z(18 - y - z)$. The partial derivative with respect to y is zero, if $z = 18 - y$. As the second derivative is negative, the function $f(y,z)$ has exactly one maximum at $y = 18 - z$ for any fixed positive number z. Define $g(z) = f(18 - z, z) = (18 - z)^2 z$. Its derivative $g'(z) = (18 - z)(18 - 3z)$ is zero at $z = 6$ ($z = 18$ does not count since then $y = 0$). As the second derivative is negative at $z = 6$, the extremum found is a maximum again. This means that out of all partial maxima of $f(y,z)$, the one for $z = 6$ is the largest. As $g(6) = f(12,6) = 864$, the initial system of equations can be satisfied only if $y = 12$ and $z = 6$. Substituting these values into the initial system leads to $3x = 84$, $72x - u^2 = 2016$. The only solution of this system is $(x,u) = (28,0)$.

O17. *(Seniors.)* The bisector of the exterior angle at vertex C of the triangle ABC intersects the bisector of the interior angle at vertex B in point K. Consider the diameter of the circumcircle of the triangle BCK whose one endpoint is K. Prove that A lies on this diameter.

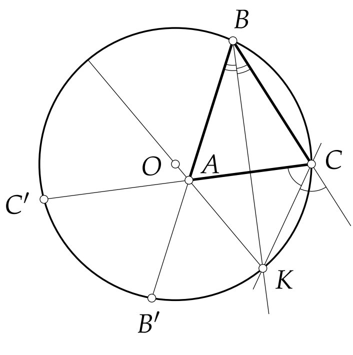


Fig. 9

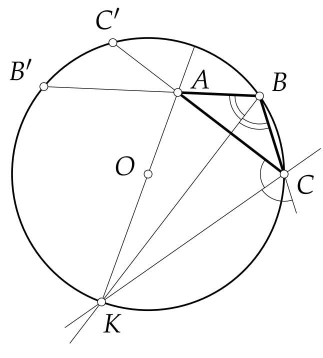


Fig. 10

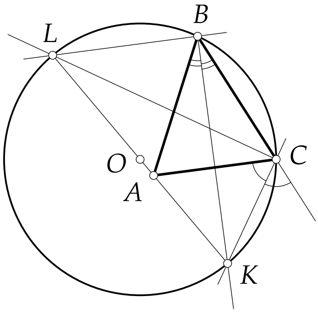


Fig. 11

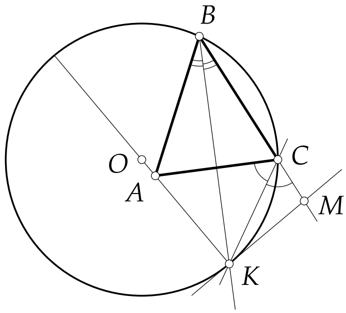


Fig. 12

Solution 1. Let B' and C' be respectively the second intersection points of the lines AB and AC with the circumcircle of the triangle BCK (Fig. 9). Notice that $\angle BKC = 180^\circ - \angle CBK - \angle BCK = 180^\circ - \frac{\angle ABC}{2} - (90^\circ + \frac{\angle ACB}{2}) = 90^\circ - \frac{\angle ABC}{2} - \frac{\angle ACB}{2}$. Hence the central angle supported by the arc BC has the size $180^\circ - \angle ABC - \angle ACB$. The central angle supported by the arc $B'C$ has the size $2\angle ABC$ and the central angle supported by the arc $C'B$ has the size $2\angle ACB$. Hence the central angle supported by the arc $B'C'$ has the size $360^\circ - (180^\circ - \angle ABC - \angle ACB) - 2\angle ABC - 2\angle ACB$, which equals $180^\circ - \angle ABC - \angle ACB$. Hence the arcs BC and $B'C'$ have the same size. As K is the midpoint of the arc $B'C$, the point B' is the reflection of the point C by the diameter drawn from point K, and C' is the reflection of point B. Hence the intersection point A of BB' and CC' has to lie on the diameter drawn from the point K.

Remark. Among the used central angles there may also be angles of size greater than 180° (reflex angles), such as the angle corresponding to the arc $B'C$ in Fig. 10.

Solution 2. Let L be the second endpoint of the diameter through K of the circumcircle of the triangle BCK (Fig. 11). Then $\angle KCL = 90^\circ$, thus CL is the bisector of the interior angle at vertex C of the triangle ABC, and $\angle KBL = 90^\circ$, so BL is the bisector of the exterior angle at vertex B of ABC.

The bisectors of the exterior angles at some two vertices of a triangle and the bisector of the interior angle at the third vertex of the triangle intersect at a common point. Thus the bisector of the exterior angle at vertex A of the triangle ABC passes through points K and L. Thus point A lies on KL.

Solution 3. Let us denote the angles of the triangle ABC at vertices A, B and C respectively α, β and γ. Let the intersection point of the line BC and the tangent to the circumcircle of triangle BCK at point K be M (Fig. 12). By the property of inscribed angles, $\angle MKC = \angle MBK = \frac{\beta}{2}$. The bisectors of the exterior angles at some two vertices of a triangle and the bisector of the interior angle at the third vertex of the triangle have a common point. Hence AK is the bisector of the exterior angle at vertex A of the triangle ABC.

Thus $\angle CKA = 180^\circ - \angle KAC - \angle KCA = 180^\circ - \frac{180^\circ - \alpha}{2} - \frac{180^\circ - \gamma}{2} = \frac{\alpha}{2} + \frac{\gamma}{2}$. Hence $\angle MKA = \angle MKC + \angle CKA = \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^\circ$. As a tangent of a circle is perpendicular to the diameter drawn from the point of tangency we have that A lies on the diameter drawn from point K.

O18. *(Seniors.)* Let n and m be positive integers. What is the biggest number of points that can be marked in the vertices of the squares of the $n \times m$ grid in such a way that no three of the marked points lie in the vertices of any right-angled triangle?

Answer: $n + m$.

Solution. All vertices of squares lie on $n + 1$ horizontal and $m + 1$ vertical lines. Suppose that at least $n + m + 1$ points are marked in the grid. Because $m > 0$, the number of marked points is greater than $n + 1$. Hence by the pigeonhole principle, at least one horizontal line contains more than one marked point. Hence at most n marked points are alone on their horizontal lines. Similarly, at most m marked points are alone on their vertical lines. Thus there exists a marked point that lies neither alone on its horizontal line nor alone on its vertical line. But then there is a right-angled triangle with vertices at marked points. So at most $n + m$ points can be marked according to the conditions of the problem.

By marking all vertices of squares of the grid that lie at the left and lower edge of the grid except for the lower left corner we have marked exactly $n + m$ points (Fig. 13 depicts the choice for a 5×7 grid). Any three of the marked points either lie on a common line or are the vertices of an obtuse triangle, so the construction satisfies the conditions of the problem.

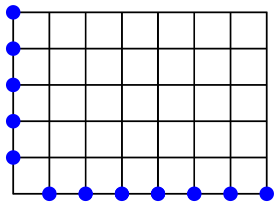


Fig. 13

Selected Problems from the Final Round of National Olympiad

F1. *(Grade 9.)* Do there exist distinct positive integers x and y such that the number $x + y$ is divisible by 2016, the number $x - y$ is divisible by 2017 and the number xy is divisible by 2018?

Answer: Yes.

Solution. For example, numbers $x = 2016 \cdot 2015 - 2018$ and $y = 2018$ meet the conditions. As $2016 \cdot 2015 > 4 \cdot 1009 = 2 \cdot 2018$ implies $x > y$, they are distinct. The sum $2016 \cdot 2015$ is divisible by 2016 and the product is obviously divisible by 2018. Furthermore, $x - y = 2016 \cdot 2015 - 2 \cdot 2018 = (2017 - 1)(2017 - 2) - 2 \cdot (2017 + 1) = 2017^2 - 3 \cdot 2017 + 2 - 2 \cdot 2017 - 2 = 2017 \cdot 2012$, whence the difference of these numbers is divisible by 2017.

Remark. This choice of the numbers is not the only possible. One can prove that all suitable numbers are of the form $x = 2016k - 2018m$ and $y = 2018m$, where k and m are integers such that $k + 2m$ is divisible by 2017. Indeed, as $2018 = 2 \cdot 1009$ and 1009 is prime, one of x and y must be divisible by 1009. Also one of x and y must be even, but as $x + y$ is divisible by the even number 2016, both must be even. Consequently, one of these numbers must be divisible by 2018. Let w.l.o.g. $y = 2018m$. As $x + y = 2016k$, we must have $x = 2016k - 2018m$. Then $x - y = 2016k - 2 \cdot 2018m = 2017(k - 2m) - (k + 2m)$, whence $x - y$ is divisible by 2017 if and only if $k + 2m$ is divisible by 2017. The pair in the solution is obtained by taking $k = 2015$ and $m = 1$.

F2. *(Grade 9.)* Find all solutions of the equation $a + b + c = 61$ in natural numbers that satisfy $\gcd(a, b) = 2$, $\gcd(b, c) = 3$, and $\gcd(c, a) = 5$.

Answer: $a = 10$, $b = 6$, $c = 45$; $a = 10$, $b = 36$, $c = 15$; $a = 40$, $b = 6$, $c = 15$.

Solution. As $\gcd(a, b) = 2$, $\gcd(b, c) = 3$ and $\gcd(c, a) = 5$, the number a is divisible by both 2 and 5, the number b is divisible by both 2 and 3, and the number c is divisible by both 3 and 5. Hence a is divisible by 10, b is divisible by 6 and c is divisible by 15. As 61 gives remainder 1 when divided by each of 2, 3 and 5, the numbers a, b and c must give remainder 1 when divided by 3, 5 and 2, respectively. Since $a, b, c \leqslant 61$, the possibilities are $a = 10$ or $a = 40$, $b = 6$ or $b = 36$, and $c = 15$ or $c = 45$. The sum 61 appears in three cases: $a = 10$, $b = 6$, $c = 45$; $a = 10$, $b = 36$, $c = 15$; $a = 40$, $b = 6$, $c = 15$. A straightforward check shows that the conditions $\gcd(a, b) = 2$, $\gcd(b, c) = 3$ and $\gcd(c, a) = 5$ are also met in all these cases.

F3. *(Grade 9.)* Find all positive integers n such that a square can be cut into n square pieces.

Answer: 1, 4 and all natural numbers starting from 6.

Solution. A partition of a square into 1 square is trivial. If $n \geqslant 2$ then a partition into $2n$ squares can be obtained by cutting n squares of side length $\frac{1}{n}$ from one side of the square and $n - 1$ more squares of the same size from a neighbouring side. One square of side length $\frac{n-1}{n}$ of the side length of the big square is left (Fig. 14 depicts the situation in the case $n = 4$ that provides a partition into 8 squares). For each $n \geqslant 2$, one can obtain a partition into $2n + 3$ squares by splitting one square in a partition into $2n$ squares into four. Thus there exist partitions into 1, 4 and every natural number starting from 6.

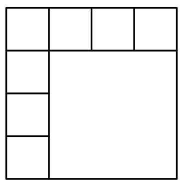


Fig. 14

It remains to show that there are no partitions of a square into 2, 3 and 5 squares. A square has 4 vertices and each vertex belongs to only one square in a partition. If two vertices belonged to the same square in the partition, this piece should be as large as the initial square, which is possi-

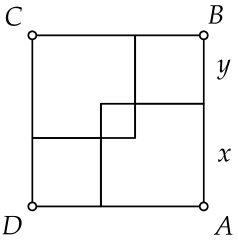


Fig. 15

ble only in partition into 1. Hence the number of squares in a partition into a larger number of squares must be at least 4. In a hypothetical partition into 5 squares, at least three sides of the initial square should adjoin exactly 2 squares in the partition. Let the initial square be $ABCD$ and let the sides DA, AB and BC adjoin exactly 2 squares in the partition. Let the squares adjoining the side AB have side lengths x and y in the order from vertex A to vertex B. The square of side length x is adjoining also the side AD and the square of side length y is adjoining the side BC, whence the other squares adjoining the sides AD and BC have side lengths y and x, respectively. If $x > y$ then the squares located by vertices A and C would overlap (Fig. 15). If $x < y$ then the squares located by vertices B and D would overlap. If $x = y$, four squares in the partition would cover the whole initial square and the fifth square cannot exist.

F4. *(Grade 9.)* Triangle ABC has $AC = BC$. The bisector of angle CAB meets side BC at point D. The difference of the sizes of some two internal angles of triangle ABD is 40°. Find all possibilities of what the size of angle ACB can be.

Answer: $68^\circ, 40^\circ, 20^\circ$ and 4°.

Solution. Denote $\angle BAC = \angle ABC = \alpha$, then $\angle ACB = 180^\circ - 2\alpha$ (Fig. 16). The sizes of the internal angles of triangle ABD are $\angle BAD = \frac{\alpha}{2}$, $\angle DBA = \alpha$ and $\angle ADB = 180^\circ - \frac{3}{2}\alpha$. Consider all cases of which two angles can have 40° as the difference of sizes.

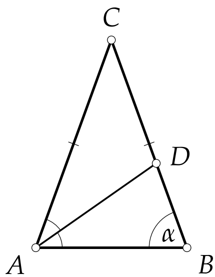


Fig. 16

- If $\alpha - \frac{\alpha}{2} = 40^\circ$ then $\alpha = 80^\circ$, whence $\angle ACB = 20^\circ$.
- The case $\frac{\alpha}{2} - \alpha = 40^\circ$ is impossible since it would imply $\alpha < 0^\circ$.
- If $\left(180^\circ - \frac{3}{2}\alpha\right) - \frac{\alpha}{2} = 40^\circ$ then $\alpha = 70^\circ$, whence $\angle ACB = 40^\circ$.
- If $\frac{\alpha}{2} - \left(180^\circ - \frac{3}{2}\alpha\right) = 40^\circ$ then $\alpha = 110^\circ$, but the base angle of an isosceles triangle cannot be obtuse.
- If $\left(180^\circ - \frac{3}{2}\alpha\right) - \alpha = 40^\circ$ then $\alpha = 56^\circ$, whence $\angle ACB = 68^\circ$.
- If $\alpha - \left(180^\circ - \frac{3}{2}\alpha\right) = 40^\circ$ then $\alpha = 88^\circ$, whence $\angle ACB = 4^\circ$.

F5. *(Grade 10.)* In the mathematics circle, Juku raised a hypothesis that, for every integer $n > 4$, at least one out of the two largest integers that are less than $\frac{n}{2}$ is relatively prime to n. Is Juku's hypothesis valid?

Answer: Yes.

Solution. If n is odd then the largest integer that is less than $\frac{n}{2}$ is $\frac{n-1}{2}$. Let d be a common divisor of numbers $\frac{n-1}{2}$ and n. Then d is a common divisor of numbers $n-1$ and n, implying that $d=1$. Hence $\frac{n-1}{2}$ and n are relatively prime, meaning that the hypothesis holds in the case of odd numbers.

If n is even then the two largest integers that are less than $\frac{n}{2}$ are $\frac{n}{2}-1$ and $\frac{n}{2}-2$. Let d_1 be a common divisor of numbers $\frac{n}{2}-1$ and n, and let d_2 be a common divisor of numbers $\frac{n}{2}-2$ and n. Then d_1 is a common divisor of numbers $n-2$ and n, and d_2 is a common divisor of numbers $n-4$ and n. Hence d_1 divides 2, i.e., is either 1 or 2, and d_2 divides 4, i.e., is either 1 or 2 or 4. If d_1 and d_2 were both larger than 1, they both should be even, whence their multiples $\frac{n}{2}-1$ and $\frac{n}{2}-2$ should be even. This is impossible, since $\frac{n}{2}-1$ and $\frac{n}{2}-2$ are consecutive integers. The contradiction shows that at least one of the divisors d_1 and d_2 equals 1. Thus one of $\frac{n}{2}-1$ and $\frac{n}{2}-2$ is relatively prime to n, meaning that the hypothesis holds in the case of even numbers, too.

Remark. The solution does not use the assumption $n>4$. The purpose of this assumption is to exclude cases that would require dealing with questions concerning the greatest common divisor of n and zero or a negative number.

F6. *(Grade 10.)* Around each vertex of a regular hexagon of side length $\sqrt{3}$ in a plane, one draws a circle of radius 1 with centre at that vertex and paints the region inside the circle blue. Find the area of the part of the plane that is painted blue.

Answer: $4\pi+3\sqrt{3}$.

Solution. The circles drawn around two neighbouring vertices of the hexagon intersect, since $2\cdot 1>\sqrt{3}$. Hence every two neighbouring circles have a common region of the shape of a lens. Since a regular hexagon can be put together from six equilateral triangles, the circumradius of the hexagon equals $\sqrt{3}$. The altitude of one equilateral triangle is $\sqrt{(\sqrt{3})^2-(\frac{\sqrt{3}}{2})^2}=\frac{3}{2}$. The distance between a vertex and the second one counting from that vertex along the circumference is 3, since it equals twice the altitude of the equilateral triangle (Fig. 17). As $2\cdot 1<3$, circles drawn around such two vertices do not intersect. Thus the circles drawn around opposite vertices do not intersect either.

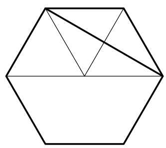


Fig. 17

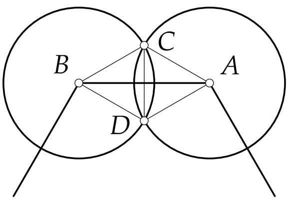


Fig. 18

The sum of the areas of all six blue circles is 6π. One must discount the area of six lenses. Let A and B be neighbouring vertices of the hexagon and let C and D be the points of intersection of the circles drawn around A and B (Fig. 18). Subtracting the area of triangle ACD from the area of sector ACD gives precisely one half of the area of a lens. We have $\left(\frac{CD}{2}\right)^2+$

$\left(\frac{AB}{2}\right)^2 = AC^2$, i.e., $\left(\frac{CD}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1^2$, whence $CD = 1$, implying that the triangle ACD is equilateral. Thus the area of sector ACD and triangle ACD equal $\frac{1}{6}\pi$ and $\frac{\sqrt{3}}{4}$, respectively. The area of the region painted blue is $6\pi - 12\left(\frac{1}{6}\pi - \frac{\sqrt{3}}{4}\right)$, which equals $4\pi + 3\sqrt{3}$.

F7. *(Grade 10.)* Find the number of solutions of the equation $|a - b| = |b - c|$ in integers from 0 to 36.

Answer: 2017.

Solution. The equation $|a - b| = |b - c|$ is satisfied if and only if either $a - b = b - c$ or $a - b = c - b$. The first equality is equivalent to $a + c = 2b$, the second is equivalent to $a = c$.

To fulfill the condition $a + c = 2b$, the numbers a and c must have the same parity. Then their sum is even and b lies between a and c, whence it also falls between the required bounds. There are 19 possibilities for choosing a or c as an even number (0, 2, ..., 36 are suitable) which gives 361 possibilities in total. The number of possibilities to choose an odd number a or c is 18 (1, 3, ..., 35 are suitable) which gives 324 possibilities in total. There are therefore 685 triples satisfying $a + c = 2b$. The number of triples satisfying $a = c$ is 1369, since a and b can be chosen arbitrarily. There are 37 solutions that satisfy both $a + c = 2b$ and $a = c$ and hence being counted twice, since these two conditions hold simultaneously if and only if $a = b = c$. Consequently, the total number of solutions meeting the conditions of the problem is $685 + 1369 - 37$ which equals 2017.

F8. *(Grade 10.)* a) The general form $\overline{ABC}$ of a three-digit number is initially written on a blackboard. Ann and Enn replace by turns letters with digits, exactly one at a time, with Ann starting. Can Ann write digits in such a way that, irrespectively of Enn's move, the resulting three-digit number would be divisible by 11? (Different letters may be replaced with equal digits but the letter A must not be replaced with zero.)

b) Ann and Enn got bored with writing the general form of the number again at the beginning of each game, and decided to change the rules as follows. First, Ann writes one digit to the blackboard, then Enn writes the second digit either to the right or to the left of it, and finally Ann completes the number with writing the third digit either to the left or to the right of the two digits already on the blackboard (writing between the digits is not allowed). Can Ann write digits in such a way that, irrespectively of Enn's move, the result would be a three-digit number (i.e., not starting with 0) that is divisible by 11?

Answer: a) No; b) Yes.

Solution. a) Let Ann replace on her first move one of letters B or C with a digit k. If Enn now replaces the other one of B and C with digit k, too, the resulting number $\overline{Akk}$ is divisible by 11 if and only if $\overline{A00}$ is divisible by 11, which in turn is the case if A is divisible by 11. The only such digit is 0 but A cannot be replaced with 0. If Ann replaces on her first move the letter A with a non-zero digit k then Enn can replace B with the digit $k-1$. The number resulting from Ann's second move differs from the number $\overline{kk0}$ by less than 11, whence it cannot be divisible by 11.

b) Let Ann write 9 on her first move. If Enn now writes before or after it a digit k and Ann writes after or before of it, respectively, the digit $9-k$, then the resulting number is divisible by 11. Ann cannot make her last move in such a way only if Enn on his move has written either 9 to the end of the number or 0 to the beginning of the number. If Enn has written 9 to the end of the number, the blackboard contains digits 99 and Ann can construct a multiple of 11 by writing 0 to the end. If Enn has written 0 to the beginning of the number, Ann can write 2 to the very beginning which results in 209, again a multiple of 11.

Remark. Similarly to the proof presented here, one can show that Ann can win after writing any digit $n \geqslant 2$ on her first move.

F9. *(Grade 11.)* Find the least positive integer n for which there exists a positive integer a such that both a and $a+735$ have exactly n positive divisors.

Answer: 4.

Solution. If numbers a and $a+735$ had exactly 2 divisors, they would be primes. These numbers are of different parity, whence one of them is even. But if $a=2$ then $a+735=737=11\cdot 67$.

If numbers a and $a+735$ had exactly 3 divisors, each of them would be a square of a prime. Similarly to the previous case, we get $a=4$. But then $a+735=739$, and 739 is not a perfect square.

On the other hand, numbers 10 and $10+735$ have exactly 4 divisors.

Remark. The case of 3 divisors can be handled also as follows. If numbers a and $a+735$ had exactly 3 divisors, they would be squares of primes, implying $735=p^2-q^2=(p-q)(p+q)$ for some primes p and q. As $735\equiv 3 \pmod 4$, one of the factors $p-q$ and $p+q$ must be congruent to 3 and the other one congruent to 1 modulo 4. But then their sum $2p$ is divisible by 4, whence p is even and cannot be the larger among two primes.

F10. *(Grade 11.)* Find all quadruples (a,b,c,d) of integers satisfying the system of equations $-a^2+b^2+c^2+d^2=1$, $3a+b+c+d=1$.

Answer: $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,0,-1,-1)$, $(1,-1,0,-1)$, and $(1,-1,-1,0)$.

Solution 1. Note that $-a^2+b^2+c^2+d^2=-2a(2a+b+c+d)+(a+b)^2+(a+c)^2+(a+d)^2$. By the second equation, $2a+b+c+d=1-a$, so the first equation reduces to $2a(a-1)+(a+b)^2+(a+c)^2+(a+d)^2=1$. If $a>1$ or $a<0$ then $2a(a-1)$ is a positive even number, whence the l.h.s. of the last equation is greater than 1. If $a=0$ then according to the first equation exactly one number among b^2, c^2 and d^2 is equal to 1, others are zero. Value -1 together with zeros would not satisfy the second equality, thus one of b, c and d must be 1, while others are zero. If $a = 1$ then according to the first equation exactly two numbers among b^2, c^2 and d^2 are equal to 1 and the remaining number is zero. The second equation is satisfied only if both non-zero variables take the value -1.

Solution 2. The given system of equations is equivalent to

$$\begin{cases} a^2 = b^2 + c^2 + d^2 - 1, \\ 3a = 1 - b - c - d. \end{cases}$$

Squaring the second equation and subtracting the first equation multiplied by 9, we get $8b^2 + 8c^2 + 8d^2 - 10 + 2b + 2c + 2d - 2bc - 2bd - 2cd = 0$, which is equivalent to $5b^2 + 5c^2 + 5d^2 + (b+1)^2 + (c+1)^2 + (d+1)^2 + (b-c)^2 + (b-d)^2 + (c-d)^2 = 13$. In the l.h.s., the values of all monomials are non-negative integers. Thus at least one number among b, c, d must be zero, otherwise the first three monomials would sum up to more than the r.h.s. W.l.o.g., assume $b = 0$, the last equation then reduces to

$$6c^2 + 6d^2 + (c+1)^2 + (d+1)^2 + (c-d)^2 = 12.$$

Obviously $|c| \leqslant 1$, $|d| \leqslant 1$, otherwise either of the first two monomials alone would be larger than the r.h.s. If $|c| = |d| = 1$, the last equation can hold only if $c + 1 = d + 1 = c - d = 0$, whence $c = d = -1$. Substituting $b = 0$ and $c = d = -1$ into the second equation of the initial system, we obtain $a = 1$. If, for example, $c = 0$, then the last equation reduces to $7d^2 + (d+1)^2 = 11$, whence $d = 1$ is the only possibility. Substituting $b = c = 0$ and $d = 1$ to the second equation, we obtain $a = 0$.

Tuples $a = 1$, $b = 0$, $c = d = -1$ and $a = 0$, $b = c = 0$, $d = 1$ together with those obtained by arbitrarily permuting the values of b, c, d satisfy also the first equation of the initial system.

Solution 3. If $a = 0$ then the first equation of the system reduces to $b^2 + c^2 + d^2 = 1$, which can hold in integers only if two numbers among b, c, d are zeros. If, w.l.o.g., $b = c = 0$, then the second equation of the system gives $d = 1$. Assume in the following that $a \neq 0$. The first equation of the system is equivalent to $b^2 + c^2 + d^2 = 1 + a^2$, which implies $b^2 \leqslant 1 + a^2 < (1 + |a|)^2$. Hence $|b| < 1 + |a|$ which implies $|b| \leqslant |a|$. Analogously we get also $|c| \leqslant |a|$ and $|d| \leqslant |a|$. Thus $|3a| = 3|a| \geqslant |b| + |c| + |d|$. The second equation of the system, however, gives $1 = 3a + b + c + d = |3a + b + c + d| \geqslant |3a| - (|b| + |c| + |d|)$. Hence either $|3a| - (|b| + |c| + |d|) = 0$ or $|3a| - (|b| + |c| + |d|) = 1$. In the first case, we get $|a| = |b| = |c| = |d|$ which does not satisfy the initial system. In the second case, assume w.l.o.g. that $|a| = |b| = |c|$ and $|d| = |a| - 1$. Substituting $|a| = |b| = |c|$ into the first equation of the initial system, we get $a^2 + d^2 = 1$, whence $|a| = 1$ and $d = 0$. If $a = 1$, the second equation implies $b = c = -1$. The case $a = -1$ does not give solutions.

Remark. After obtaining the inequalities $|b| \leqslant |a|$, $|c| \leqslant |a|$, $|d| \leqslant |a|$ one can also argue as follows. If $a < 0$ then $3a + b + c + d = -3|a| + b + c + d \leqslant -3|a| + |b| + |c| + |d| \leqslant 0$, which contradicts the second equation. Consequently $a > 0$. Thus $a + b = |a| + b \geqslant |a| - |b| \geqslant 0$ and, analogously, $a + c \geqslant 0$ and $a + d \geqslant 0$. As the second equation of the initial system is equivalent to $(a + b) + (a + c) + (a + d) = 1$, one number among $a + b$, $a + c$, $a + d$ must be 1, while others are zeros. If $a + b = 1$ and $c = d = -a$ then the first equation reduces to $a^2 + b^2 = 1$, implying $a = 1$, $b = 0$, $c = d = -1$. The other two solutions are obtained analogously.

Solution 4. We rewrite the given equations as $b^2 + c^2 + d^2 = 1 + a^2$, $b + c + d = 1 - 3a$. We now apply the AM-QM inequality to the numbers $|b|$, $|c|$ and $|d|$ to obtain

$$\sqrt{\frac{1+a^2}{3}} = \sqrt{\frac{b^2+c^2+d^2}{3}} \geqslant \frac{|b|+|c|+|d|}{3} \geqslant \frac{|b+c+d|}{3} = \frac{|1-3a|}{3}.$$

Thus $\frac{1+a^2}{3} \geqslant \frac{(1-3a)^2}{9}$, whence $3 + 3a^2 \geqslant 1 - 6a + 9a^2$, implying $6a^2 - 6a - 2 \leqslant 0$. Solving the quadratic inequality gives $\frac{3-\sqrt{21}}{6} \leqslant a \leqslant \frac{3+\sqrt{21}}{6}$, whence $a = 0$ or $a = 1$. Now proceed as in Solution 1.

F11. *(Grade 11.)* Let ABC be a scalene triangle with median AM. Let K be the point of tangency of the incircle of triangle ABC with the side BC. Prove that if the length of the side BC is the arithmetic mean of the lengths of the sides AB and AC then the bisector of the angle BAC passes through the midpoint of the line segment KM.

Solution. Let N be the intersection point of the bisector of angle BAC and side BC; it suffices to prove that $KN = MN$ (Fig. 19). The bisector property implies $\frac{NC}{NB} = \frac{AC}{AB}$. Substituting $NC = BC - NB$ gives

$$NB = \frac{BC}{1 + \frac{AC}{AB}} = \frac{AB \cdot BC}{AB + AC}.$$

As $AB + AC = 2BC$ by assumption, this implies $NB = \frac{AB}{2}$.

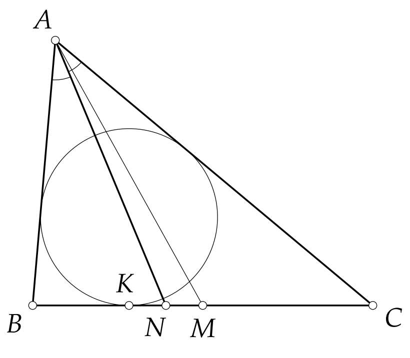


Fig. 19

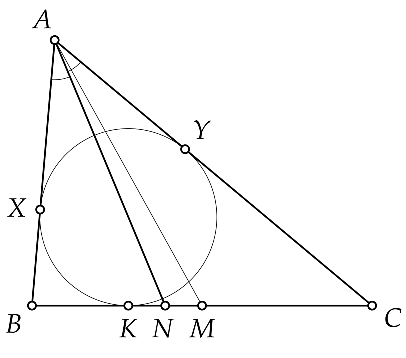


Fig. 20

On the other hand, let X and Y be the points of tangency of the incircle of triangle ABC with sides AB and AC, respectively (Fig. 20). Then $AX = AY$, $BX = BK$ and $CK = CY$, whence $BC = BK + CK = BX + CY = AB - AX + AC - AY = AB + AC - 2AX = 2BC - 2AX$. Thus $BC = 2AX$, implying $AX = BM$. Consequently, $KN = BN - BK = BN - BX = \frac{AB}{2} - (AB - AX) = AX - \frac{AB}{2} = BM - \frac{AB}{2} = BM - BN = MN$.

F12. *(Grade 11.)* Call a tuple $(a_1, \dots, a_n)$ of real numbers *stable* if the sums $a_1 + a_2 + \ldots + a_k$, as well as the sums $a_k + a_{k+1} + \ldots + a_n$, where in both cases $0 < k \leqslant n$, are either all negative or all non-negative.

For instance, the tuple $(3, -1, 2)$ is stable, since:

$$\begin{array}{rcl@{\qquad}rcl} 3 & \geqslant & 0, & 2 & \geqslant & 0, \\ 3 + (-1) & \geqslant & 0, & (-1) + 2 & \geqslant & 0, \\ 3 + (-1) + 2 & \geqslant & 0; & 3 + (-1) + 2 & \geqslant & 0. \end{array}$$

Prove that in any stable tuple with at least 3 terms where all terms are alternately negative and non-negative (it is unknown whether the first term is negative or non-negative), there exist 3 consecutive terms that together (without reordering) form a stable tuple on their own.

Solution. Consider terms whose absolute value is minimal in the tuple. If there exists a negative such element, denote it a_i, then the sum of a_i and its any neighbour is non-negative. Thus a_i is neither the first nor the last in the tuple because of stability of the tuple. But then both $a_{i-1} + a_i$ and $a_i + a_{i+1}$ are non-negative, as well as $a_{i-1} + a_i + a_{i+1}$, hence a_{i-1}, a_i, a_{i+1} together form a stable subtuple. If all elements with minimal absolute value are non-negative then let a_i be any of them. Analogously to the previous case, both $a_{i-1} + a_i$ and $a_i + a_{i+1}$ are negative, as well as $a_{i-1} + a_i + a_{i+1}$, whence a_{i-1}, a_i, a_{i+1} together form a stable tuple.

Remark. Several less elegant approaches are possible. First, one can prove by induction that if the claim didn't hold then terms of the tuple $(a_1 + a_2, a_2 + a_3, \dots, a_{n-1} + a_n)$ would be alternately negative and non-negative, whereby $a_1 + a_2$ would have the same sign as a_1 and $a_{n-1} + a_n$ would have the same sign as a_n. This would imply that the numbers of terms in tuples $(a_1, a_2, \dots, a_n)$ and $(a_1 + a_2, a_2 + a_3, \dots, a_{n-1} + a_n)$ would have the same parity which is impossible.

Second, one can prove that $|a_1| \geqslant |a_2|$ and $|a_{n-1}| \leqslant |a_n|$, whereby these inequalities are strict in the case $a_1 < 0$. Furthermore, one can prove that a triple (x, y, z) is stable if and only if $|x| \geqslant |y|$ and $|z| \geqslant |y|$, with strict inequalities in the case $x < 0$. From these facts, one can prove that if the claim doesn't hold then $|a_1| > |a_2| \geqslant |a_3| \geqslant \ldots \geqslant |a_{n-1}| \geqslant |a_n|$ if $a_1 < 0$ and $|a_1| \geqslant |a_2| > |a_3| > \ldots > |a_{n-1}| > |a_n|$ if $a_1 \geqslant 0$. In both cases, we get a contradiction with the magnitude relationship between a_{n-1} and a_n.

F13. *(Grade 11.)* Given positive integers a, b, c and d and

$$(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) = u,$$
$$ab+ac+ad+bc+bd+cd = v,$$

prove that the product uv is divisible by 3.

Solution. If among numbers a, b, c, d there are two that give either remainders 0 and 0 or remainders 1 and 2 modulo 3, the sum of these two numbers is divisible by 3. Hence u, as well as uv, is divisible by 3. Now study the case where at most one among the numbers a, b, c, d is divisible by 3 and all numbers not divisible by 3 are congruent modulo 3. If exactly one among numbers a, b, c, d is divisible by 3 then the products of this number with all other numbers are divisible by 3. Other numbers form 3 pairs whose products of components are congruent modulo 3. Hence the sum v of all six pairwise products is divisible by 3. If none of a, b, c, d is divisible by 3 then the pairwise products are all congruent modulo 3. Again, as the number of pairs is divisible by 3, this implies that the sum v of the products is divisible by 3. Consequently, uv is divisible by 3 in this case, too.

F14. *(Grade 12.)* Positive integer b is obtained by reordering the digits in a positive integer a. Which of the following claims are definitely true?

a) The sums of the digits of numbers $2a$ and $2b$ are equal.
b) The sums of the digits of numbers $3a$ and $3b$ are equal.
c) The sums of the digits of numbers $5a$ and $5b$ are equal.

Answer: a) and c).

Solution. Call digits 0, 1, 2, 3, 4 *small* and digits 5, 6, 7, 8, 9 *large*. Denote the digits of k-digit number n from right to left by $(n)_0, (n)_1, \ldots, (n)_{k-1}$. Denote by $\Sigma(n)$ the sum of all digits of n and by $l(n)$ the number of large digits of n.

a) If $(a)_i$ is small then $(2a)_i = 2(a)_i$ or $(2a)_i = 2(a)_i + 1$ depending on whether $(a)_{i-1}$ is small or large. Similarly if $(a)_i$ is large then $(2a)_i = 2(a)_i - 10$ or $(2a)_i = 2(a)_i - 9$ depending on whether $(a)_{i-1}$ is small or large. In other words, when multiplying by 2, each large digit necessitates decreasing of the digit at its place by 10 and increasing of the preceding digit by 1 in comparison with a small digit. Hence, for every natural number n, $\Sigma(2n) = 2\Sigma(n) - 9l(n)$. As $\Sigma(a) = \Sigma(b)$ and $l(a) = l(b)$, we have $\Sigma(2a) = \Sigma(2b)$.

b) If $a = 34$ and $b = 43$ then $\Sigma(3a) = 1+0+2 = 3$ but $\Sigma(3b) = 1+2+9 = 12$.

c) Obviously $\Sigma(10a) = \Sigma(10b)$. By part a), on the other hand, $\Sigma(10n) = \Sigma(2 \cdot 5n) = 2\Sigma(5n) - 9l(5n)$, holding for every natural number n. Hence $2\Sigma(5a) - 9l(5a) = 2\Sigma(5b) - 9l(5b)$. The digit $(5n)_i$ is large if and only if the digit $(n)_i$ is odd, because a carry during multiplying by 5 can be at most 4. Thus $l(5n)$ is the number of odd digits of n, implying $l(5a) = l(5b)$. Consequently, $\Sigma(5a) = \Sigma(5b)$.

F15. *(Grade 12.)* Real numbers x, y and z satisfy $x+y+z=4$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{3}$. Find the largest and the smallest possible value of the expression $x^3+y^3+z^3+xyz$.

Answer: 64 is both the largest and the smallest.

Solution. Note that

$$(x+y+z)^3 = x^3+y^3+z^3+3(x^2y+xy^2+x^2y+xy^2+y^2z+yz^2)+6xyz,$$

while

$$3(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)xyz = 3(x+y+z)(xy+xz+yz)$$
$$= 3(x^2y+xy^2+x^2z+xz^2+y^2z+yz^2)+9xyz.$$

Thus

$$(x+y+z)^3-3(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)xyz = x^3+y^3+z^3-3xyz.$$

By assumptions, $x+y+z=4$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{3}$. Hence $64-4xyz = x^3+y^3+z^3-3xyz$, implying $x^3+y^3+z^3+xyz=64$. Consequently, the expression $x^3+y^3+z^3+xyz$ has only one value 64.

Remark. An example of numbers satisfying the conditions is $x=1$, $y=\frac{3-3\sqrt{3}}{2}$ and $z=\frac{3+3\sqrt{3}}{2}$. Then $\frac{1}{x}=1$, $\frac{1}{y}=\frac{-1-\sqrt{3}}{3}$ and $\frac{1}{z}=\frac{-1+\sqrt{3}}{3}$.

F16. *(Grade 12.)* Prove that in every triangle there is a median whose length squared is at least $\sqrt{3}$ times the area of the triangle.

Solution 1. Assume w.l.o.g. that BC is the shortest side of the triangle. Then the least angle of the triangle is by vertex A. Denote $a=BC$, $b=CA$, $c=AB$, $\alpha=\angle BAC$ and let m be the length of the median drawn from vertex A. By assumptions made at the beginning of the solution, $\alpha \leqslant 60^\circ$. Let D, E, F be the midpoints of sides BC, CA, AB, respectively (Fig. 21). As $\angle AFD = 180^\circ - \alpha$ because of $DF \parallel CA$, the law of cosines in triangle AFD implies $m^2 = (\frac{b}{2})^2+(\frac{c}{2})^2-2\cdot\frac{b}{2}\cdot\frac{c}{2}\cos\angle AFD = \frac{b^2}{4}+\frac{c^2}{4}+\frac{bc}{2}\cos\alpha \geqslant \frac{b^2}{4}+\frac{c^2}{4}+\frac{bc}{2}\cos 60^\circ = \frac{b^2+c^2+bc}{4}$. As $b^2+c^2 \geqslant 2bc$, we obtain $m^2 \geqslant \frac{3bc}{4}$.

On the other hand, let S be the area of the triangle ABC. Then $S = \frac{1}{2}bc\sin\alpha \leqslant \frac{1}{2}bc\sin 60^\circ \leqslant \frac{\sqrt{3}bc}{4}$. Before we obtained the inequality $m^2 \geqslant \frac{3bc}{4} = \sqrt{3}\cdot\frac{\sqrt{3}bc}{4}$. Consequently, $m^2 \geqslant \sqrt{3}\,S$.

Solution 2. Let the median drawn from vertex A of the triangle ABC be AD and the centroid of the triangle be M. W.l.o.g., let AD be the longest median of triangle ABC; denote $AD=m$. Draw to both sides of the median AD triangles ADX and ADY that have right angles by vertex D and angles of size 30° by vertex A; then AXY is an equilateral triangle having median AD and centroid M in common with triangle ABC (Fig. 22). We show next that the area of triangle AXY is at least as large as the area of triangle ABC. Since a median divides the triangle into two parts of equal area, it suffices

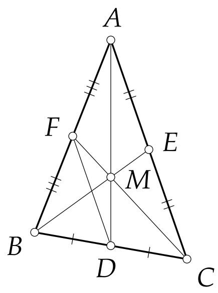


Fig. 21

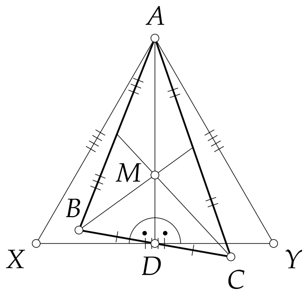


Fig. 22

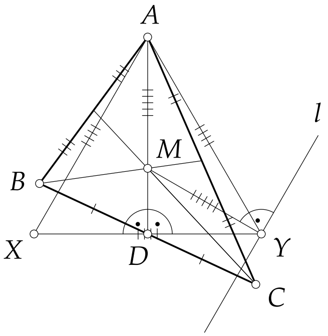


Fig. 23

to show that the area of triangle ADX is at least as large as the area of triangle ADB. If point B lies inside the triangle ADX or on its side then this claim holds obviously. If point C lies inside the triangle ADY or on its side then this claim holds by symmetry. It remains to handle the case where both B and C lie outside the triangle AXY (Fig. 23). Suppose w.l.o.g. that line segments DB and AX intersect (the other case where line segments DC and AY intersect is symmetric). Let l be the line parallel to AX passing through Y. As line l is symmetric w.r.t. point D with line AX and $DC = DB$, line segment DC intersects line l. Therefore $MC > MY = MA$, which contradicts the assumption that AD is the longest median of the triangle ABC. Hence this case cannot appear. As $XD = YD = \frac{m}{\sqrt{3}}$, the area of the triangle AXY is $m \cdot \frac{m}{\sqrt{3}} = \frac{m^2}{\sqrt{3}}$. By the argumentation above, the area of the triangle ABC must be at most $\frac{m^2}{\sqrt{3}}$.

Solution 3. Let the median of the triangle ABC drawn from point A be AD and the centroid of the triangle be M. W.l.o.g., let AD be the longest median of the triangle ABC; denote $AD = m$. Then A is the vertex of the triangle with largest distance from point M. Thus vertices B and C lie in circle c with centre M and radius MA. Let B' and C' be the second intersection points of rays AB and AC, respectively, with circle c (Fig. 24); then the area of the triangle $AB'C'$ is at least as large as the area of triangle ABC. Let X and Y be points on circle c such that triangle AXY is equilateral (Fig. 25); then the area of AXY is at least as large as the area of triangle $AB'C'$ and also at least as large as the area of triangle ABC. The triangle AXY can be divided into three equal triangles MAX, MXY and MYA whose total area is $\frac{3}{2} \cdot \left(\frac{2}{3}m\right)^2 \sin 120°$, which equals $\frac{m^2}{\sqrt{3}}$. Hence the area of the triangle ABC is at most $\frac{m^2}{\sqrt{3}}$.

Solution 4. Let M be the centroid of the triangle ABC and m be the length of its longest median. The area of the triangle ABC is

$$S = \frac{1}{2} \cdot (MB \cdot MC \cdot \sin\alpha + MC \cdot MA \cdot \sin\beta + MA \cdot MB \cdot \sin\gamma),$$

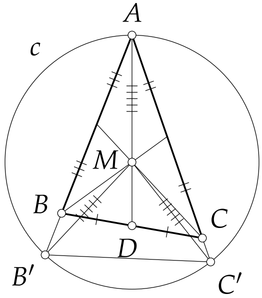


Fig. 24

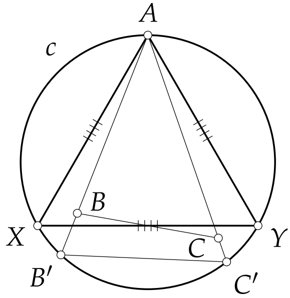


Fig. 25

where $\alpha = \angle BMC$, $\beta = \angle CMA$ and $\gamma = \angle AMB$. As $MA \leqslant \frac{2}{3}m$, $MB \leqslant \frac{2}{3}m$ and $MC \leqslant \frac{2}{3}m$, we obtain $S \leqslant \frac{1}{2} \cdot (\frac{2}{3}m)^2 \cdot (\sin\alpha + \sin\beta + \sin\gamma)$. As α, β and γ are less than 180°, Jensen's inequality applies and gives

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{3} \leqslant \sin\frac{\alpha+\beta+\gamma}{3} = \sin\frac{360^\circ}{3} = \sin 120^\circ = \frac{\sqrt{3}}{2}.$$

Consequently, $S \leqslant \frac{1}{2} \cdot \frac{4}{9}m^2 \cdot 3 \cdot \frac{\sqrt{3}}{2} = \frac{m^2}{\sqrt{3}}$, directly implying the claim.

F17. *(Grade 12.)* Inside a circle of radius 1 (or on the circumference), one marks n points in such a way that the minimal distance between two marked points is as large as possible. Let d_n be this distance between the two closest points. Is it true that $d_{n+1} < d_n$ for every natural number $n \geqslant 2$?

Answer: No.

Solution. We show that $d_6 \leqslant 1 \leqslant d_7$. For the first inequality, assume arbitrary six points A_1, A_2, A_3, A_4, A_5, A_6 being marked in the circle. Let the centre of the circle be O. If $A_i = O$ for some i, the distance between A_i and any other marked points is at most 1. Assume in the rest that $A_i = O$ for no i. Let α be the smallest angle that arises between some two rays OA_i and OA_j, where $i, j = 1, 2, 3, 4, 5, 6$ (Fig. 26). Then $\alpha \leqslant 60^\circ$, since the sum of angles between consecutive rays is 360°. If $A_iA_j > 1$ then A_iA_j is the largest side of the triangle OA_iA_j, as the lengths of OA_i and OA_j do not exceed 1. The angle opposite to the longest side is the largest, whence α should be larger than 60°, contradiction. Thus there exist two marked points at distance at most 1 from each other. As the choice of the points was arbitrary, this establishes $d_6 \leqslant 1$.

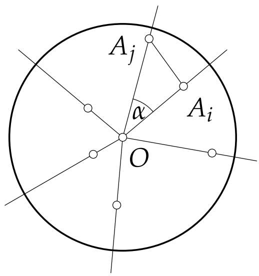


Fig. 26

On the other hand, when marking the vertices of a regular hexagon inscribed into the circle together with the centre of the circle, the distance between any two consecutive marked points on the circumference is equal to 1 and their distance from the remaining point is also 1. Hence $d_7 \geqslant 1$.

Remark. The non-strict inequality $d_{n+1} \leqslant d_n$ holds for every n. Indeed, if there is a couple with distance at most a among arbitrary n marked points then there is such a couple among arbitrary $n+1$ marked points either.

F18. *(Grade 12.)* Find all positive integers k for which the integers $1, 2, \ldots,$ 2017 can be divided into k groups in such a way that the sums of numbers in these groups are k consecutive terms of an arithmetic sequence.

Answer: $1, 2, 1009, 2017$.

Solution. Let the arithmetic sequence have the first term a and the common difference d. The sum of all terms equals the sum of numbers $1, 2, \ldots,$ 2017, i.e., $\frac{2a+(k-1)d}{2} \cdot k = \frac{2017 \cdot 2018}{2}$, whence $(2a + (k-1)d) \cdot k = 2017 \cdot 2018 =$ $2 \cdot 1009 \cdot 2017$. Thus the product $2 \cdot 1009 \cdot 2017$ is divisible by k. Since $2, 1009,$ and 2017 are primes and $k \leqslant 2017$ by assumption, the only possibilities are $k = 1$, $k = 2$, $k = 1009$, and $k = 2017$.

All these can occur indeed. A partition into 1 group trivially satisfies the conditions. An arbitrary partition into 2 groups also provides two consecutive terms of some arithmetic sequence. Coupling each even number with the next odd number, we get 1008 groups of size 2, whose sums are consecutive terms of the arithmetic sequence $5, 9, 13, \ldots$. Forming one additional group containing 1 as the only element, we obtain a partition satisfying the conditions. Finally, the conditions will also be met by the partition where every integer $1, 2, \ldots, 2017$ belongs to a separate group.

IMO Team Selection Contest I

S1. Do there exist two positive powers of 5 such that the number obtained by writing one after the other is also a power of 5?

Answer: No.

Solution. Suppose that $5^x \cdot 10^n + 5^y = 5^z$, where 5^y has n digits. Then $5^{x+n} \cdot 2^n = 5^y \cdot (5^{z-y} - 1)$, whence $2^n = 5^{z-y} - 1$. Case $n = 1$ does not work. For case $n = 2$ we get $z - y = 1$. Since 5^y has 2 digits, the only possibility is $y = 2$ and $z = 3$, whence $x = 0$, which is not positive. Case $n > 2$ yields $5^{z-y} \equiv 1 \pmod 8$, thus $z - y = 2k$ for an integer k. Now $2^n = 25^k - 1 = 24 \cdot (25^{k-1} + \ldots + 1)$, this is impossible, since $3 \mid 24$.

S2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that $\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leqslant C$.

Solution. See IMO 2016 shortlist, problem A2.

S3. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incenter. The line BI meets AC at D, and the line through D perpendicular to AC

meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Solution. See IMO 2016 shortlist, problem G4.

S4. Let ABC be an isosceles triangle with apex A and altitude AD. On AB, choose a point F distinct from B such that CF is tangent to the incircle of ABD. Suppose that ΔBCF is isosceles. Show that those conditions uniquely determine: a) which vertex of BCF is its apex; b) the size of $\angle BAC$.

Solution. a) Consider cases of the location of the vertex angle of the triangle BCF (Fig. 27).

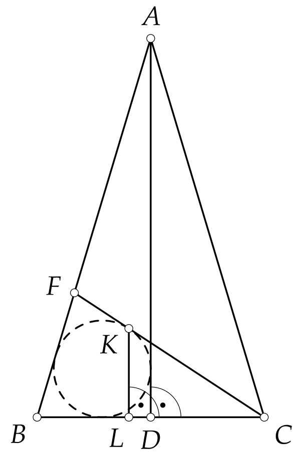


Fig. 27

If F were the apex, then F would lie on the perpendicular bisector of the side BC, i.e., on the line AD, whence $F = A$. Therefore AC would be a tangent of the incircle of the triangle ABD. But the lines AB and AD are tangents of the same circle. There can be at most two tangents drawn from one point to one circle. Hence this case is impossible.

Let B be the apex. As CBF is a base angle of the isosceles triangle ABC, we have $\angle CBF < 90^\circ$. Hence $\angle BCF > 45^\circ$. Let K be the tangent point of the line CF and the incircle of the triangle ABD and let L be the projection of the point K onto the line BC. By construction, $KL < 2r$, where r is the radius of the incircle of the triangle ABD. On the other hand, $\angle LCK = \angle BCF > 45^\circ$ implies $KL > CL > CD = BD > 2r$. Hence this case is impossible, too.

This shows that the apex of triangle BCF can only be C.

b) Let the apex be C. Fix a point D, mutually perpendicular lines l_1 and l_2 both passing through C, and circle c that is tangent to both lines; let the radius of the circle be r. Choose point A on l_1 in such a way that $DA > 2r$ and the tangent point of line l_1 and circle c lies on the line segment DA. Point B is determined by the location of A as the point of intersection of line l_2 and the second tangent line of circle c passing through A, point C is symmetric to B w.r.t. DA and F is defined as in the problem. When point A moves away from D, points B and C get closer to D, whence $\angle BAC$ decreases and $\angle BCF$ increases. Thus these angles can equal only in one case.

Remark. By extending the argument given to part b) of the problem, it can be shown that an isosceles triangle satisfying the conditions of the problem exists. It suffices to note that $\angle BAC$ can get infinitely close to zero, while $\angle BAC = 90^\circ$ implies $\angle BCF < \angle BCA = 45^\circ < \angle BAC$.

Calculations show that the size α of apex angle that fulfills the conditions of the problem satisfies the equation $\tan^3\alpha - 8\tan^2\alpha + 17\tan\alpha - 8 = 0$ and its approximate value is $\alpha \approx 33.3^\circ$.

S5. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111, and 100.) The contestant, who is allowed to look at the strings written by the deputy leader, tries to guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Solution. See IMO 2016 shortlist, problem C1.

S6. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the equation

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)(f(f(x^2)) + f(f(y^2))).$$

for all $x, y \in \mathbb{R}^+$.

Answer: $f(x) = \frac{1}{x}$.

Solution. See IMO 2016 shortlist, problem A4. (That problem was proposed by Estonia.)

IMO Team Selection Contest II

S7. Let n be a positive integer. In how many ways can an $n \times n$ table be filled with integers from 0 to 5 such that

a) the sum of each row is divisible by 2 and the sum of each column is divisible by 3;

b) the sum of each row is divisible by 2, the sum of each column is divisible by 3 and the sum of each of the two diagonals is divisible by 6?

Answer: a) 6^{n^2-n}; b) 1, if $n = 1$; 6, if $n = 2$; 6^{n^2-n-2}, if $n \geqslant 3$.

a) Let's fill the top left $(n-1) \times (n-1)$ subtable arbitrarily; this can be done in $6^{(n-1)^2}$ ways. Now there are 3 ways to fill each of the top $n-1$ cells of the rightmost column and 2 ways to fill each of the left $n-1$ cells of the bottom row to satisfy the requirements. The value for the last empty cell in the bottom right is then uniquely determined (mod 2 by the bottom row, and mod 3 by the rightmost column). In conclusion, there are $6^{(n-1)^2} \cdot 3^{n-1} \cdot 2^{n-1} = 6^{n^2-n}$ ways to fill the table.

b) For $n = 1$, the only solution is writing 0 into the single cell. For $n = 2$, let a be the top left number. The bottom right must then be $(6 - a) \bmod 6$. Using the conditions for rows and columns, for the top right number x we get the equations $x \equiv -a \pmod 2$ and $x \equiv a \pmod 3$, and for the bottom left number y, $y \equiv a \pmod 2$ and $y \equiv -a \pmod 3$. The Chinese remainder theorem determines x and y uniquely, and we see from the equations that their sum is also divisible by 6. Thus there are 6 ways to fill the table in this case, one for each value of a.

Consider now $n \geqslant 3$. Fill the top left $(n-1) \times (n-1)$ subtable arbitrarily; this can be done in $6^{(n-1)^2}$ ways. The bottom right cell's value is uniquely determined by other values on the falling diagonal. Denote the value in the top left cell by a, the sum of the 2nd to $n-1$-st cells (inclusive) in the top row by b, the sum of the 2nd to $n-1$-st cells in the leftmost column by c, and the sum of 2nd to $n-1$-st cells on the rising diagonal by d.

Using the Chinese remainder theorem, fill the top right cell with the unique value x such that $x \equiv -a-b \pmod 2$ and $x \equiv a+c-d \pmod 3$, and the bottom left cell with the unique value y such that $y \equiv a+b-d \pmod 2$ and $y \equiv -a-c \pmod 3$. The divisibility conditions are now fulfilled for the top row, the leftmost column and both diagonals (the rising diagonal is verified by summing mod 2 and mod 3 separately).

Now, we leave one cell both in the rightmost column and in the bottom row empty for the time being. For the other $n-3$ empty cells in the rightmost column, there are 3 possible values for each, and for the other $n-3$ empty cells in the bottom row, 2 values for each. Having made all those choices (which can be done in $3^{n-3} \cdot 2^{n-3}$ ways), the values for the two remaining cells are now uniquely determined (mod 2 by the values in the respective row, and mod 3 by the column). The total number of ways to fill the table is $6^{(n-1)^2} \cdot 3^{n-3} \cdot 2^{n-3} = 6^{n^2-n-2}$.

S8. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Solution. See IMO 2016 shortlist, problem A1.

S9. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point $T \in S$ such that for every point $Q \in S$, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point $A \in S$ and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) \colon x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Solution. See IMO 2016 shortlist, problem G3.

S10. Let ABC be a triangle with $AB = \frac{AC}{2} + BC$. Consider the two semicircles outside the triangle with diameters AB and BC. Let X be the orthogonal projection of A onto the common tangent line of those semicircles. Find $\angle CAX$.

Answer: 60°.

Solution. Let K and L be the midpoints of the sides AB and BC, respectively, and M and N the feet of perpendiculars from K and L, respectively, to the common tangent of the semicircles (Fig. 28). Then $\angle CAX = \angle LKM$. As KM and LN are radii of the semicircles, $KM = \frac{AB}{2}$ and $LN = \frac{BC}{2}$. Let Y be the intersection point of line KL with the common tangent of the semicircles. As triangles KYM and LYN are similar, $\frac{KY}{LY} = \frac{KM}{LN}$.

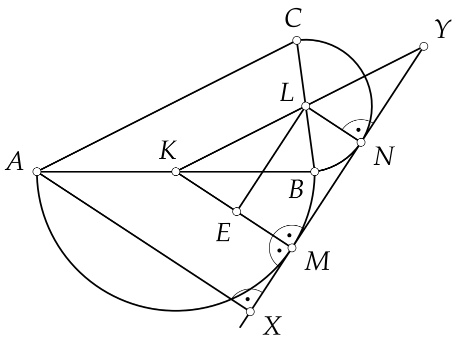


Fig. 28

Thus $\frac{KL+LY}{LY} = \frac{AB}{BC}$, whence $LY = \frac{KL \cdot BC}{AB-BC} = \frac{AC}{2} \cdot \frac{BC}{\frac{AC}{2}} = BC$. Therefore $\sin \angle NYL = \frac{LN}{LY} = \frac{\frac{BC}{2}}{BC} = \frac{1}{2}$, implying $\angle NYL = 30^\circ$. Consequently, $\angle CAX = \angle LKM = 90^\circ - \angle NYL = 60^\circ$.

S11. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and $S(P(n)) = P(S(n))$.

Solution. See IMO 2016 shortlist, problem N1.

S12. Let n be a natural number, $n \geqslant 3$. Find the maximal number of diagonals of a regular n-gon one can select in such a way that every two selected diagonals that intersect each other inside the polygon are perpendicular.

Answer: $n-2$ if n is even; $n-3$ if n is odd.

Solution. If n is odd, one can select all $n-3$ diagonals connecting one fixed vertex to others. In order to prove that the conditions of the problem do not allow more, it suffices to show that no two diagonals are perpendicular. Fix one diagonal arbitrarily; it partitions the boundary of the polygon into two halves, out of which one contains an even number of vertices and the other contains an odd number of vertices. In the latter half, the side that connects two medium vertices is parallel to the chosen diagonal. Thus the existence of two perpendicular diagonals would imply the existence of two perpendicular sides. This is possible only if $n \equiv 0 \pmod 4$, contradiction.

If $n \equiv 2 \pmod 4$, one can select every second vertex on the boundary and select initially all diagonals that connect two consecutive selected vertices. Furthermore, select all $\frac{n}{2} - 3$ diagonals connecting one fixed selected vertex to all other selected vertices that it is not connected to yet. Finally, select the diagonal connecting this very vertex to its opposite vertex of the original polygon. This way, one selects $n-2$ diagonals. If $n \equiv 0 \pmod 4$ then one can initially select $\frac{n}{2}$ diagonals as in the previous case, and then select $\frac{n}{2}-2$ more diagonals in the $\frac{n}{2}$-gon formed by the selected diagonals according to the algorithm described in this paragraph.

Now prove that selecting more diagonals is impossible. At first we show that all selected diagonals that intersect some other selected diagonals must lie in two perpendicular directions. Indeed, consider one pair of mutually intersecting diagonals. The number of vertices of the polygon lying between two endpoints of distinct diagonals under consideration is less than half of the number of all vertices of the polygon. If one more pair of mutually intersecting diagonals is added, the same holds for it. This means that the other pair cannot be fit entirely in none of the windows left there by the initial pair of diagonals. Thus at least one diagonal from the first pair and one from the second pair intersect, which means that they all must lie in two perpendicular directions.

Let there be d selected diagonals and k intersection points of selected diagonals. Consider pieces of the plane into which the selected diagonals divide the interior of the polygon; all these pieces are polygons whose vertices coincide with vertices of the original polygon and the intersection points of selected diagonals. The sum of internal angles of all pieces is $(n-2)\cdot 180^\circ + k \cdot 360^\circ$. The sum of the numbers of vertices of these pieces is $n+2d+4k$, since the initial polygon has n vertices, each diagonal adds 2 endpoints and each intersection of two diagonals adds 4. Let w be the number of pieces; then $(n-2)\cdot 180^\circ + k\cdot 360^\circ = (n+2d+4k-2w)\cdot 180^\circ$, whence $n-2+2k = n+2d+4k-2w$, implying $w = d+k+1$.

Let w' be the number of pieces with at least 4 vertices. All pieces with at least two right angles have at least 4 vertices, whence every line segment connecting two neighbouring intersection points of any diagonal is a side of such piece. Let there be a horizontal and b vertical diagonals selected, w.l.o.g. $a \leqslant b$. Then there are $k-a$ pieces whose two right angles are consecutive intersection points on some horizontal selected diagonal and which themselves lie above this diagonal. In addition, there are $b-1$ pieces whose two right angles are consecutive intersection points of some horizontal selected diagonal and which lie below this diagonal and below which there are no more horizontal diagonals. Hence $w' \geqslant k-a+b-1 \geqslant k-1$.

Consequently, $n+2d+4k \geqslant 4w' + 3(w-w') = 3w+w' \geqslant 3w+k-1 = 3(d+k+1)+k-1 = 3d+4k+2$, whence $d \leqslant n-2$.

Remark 1. In the case $n \equiv 0 \pmod 4$, another suitable construction can be given as follows. Enumerate vertices with numbers 0, 1, ..., $n-1$ in clockwise order. Select all diagonals that connect any vertex i with the vertex $\frac{n}{2}-i$ for every $i = -\frac{n}{4}+1, \ldots, \frac{n}{4}-1$ (there are $\frac{n}{2}-1$ such diagonals) and all diagonals perpendicular to them (the number of the latters is also $\frac{n}{2} - 1$). Altogether, we select $n - 2$ diagonals that all lie in two perpendicular directions. (Fig. 29 depicts the situation in the case $n = 12$).

Remark 2. This problem, proposed by Estonia, appeared as C5 in the IMO 2016 shortlist. The solution presented here appeared in a contest paper and is not given in the shortlist. For other solutions, see the shortlist.

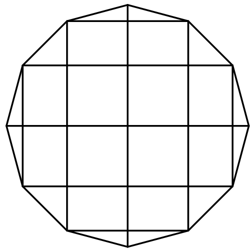


Fig. 29

Problems Listed by Topic

Number theory: O1, O5, O6, O10, O15, F1, F2, F5, F8, F9, F13, F14, F18, S1, S7, S11.
Algebra: O2, O9, O11, O14, O16, F10, F15, S2, S6, S8.
Geometry: O3, O7, O12, O17, F4, F6, F11, F16, S3, S4, S9, S10.
Discrete mathematics: O4, O8, O13, O18, F3, F7, F12, F17, S5, S12.