Estonian Math Competitions
2014/2015

The Gifted and Talented Development Centre
Tartu 2015

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WE THANK:

Estonian Ministry of Education and Research

University of Tartu

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Estonian Mathematical Olympiad

http://www.math.olympiaadid.ut.ee/

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade can participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May; in this school-year experimentally in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous olympiads are available at the Estonian Mathematical Olympiads website.

Besides the above-mentioned contests and the quiz "Kangaroo" other regional and international competitions and matches between schools are held as well.

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This booklet presents the problems of the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

O1. *(Juniors.)* We begin by writing down the number 1. On every step, under the last written number we write either that number multiplied by two, or a number obtained by shuffling the digits of the last number (not allowing numbers to begin with the digit zero). Is it possible that after a finite number of such steps, we will write down: a) the number 1000000000; b) the number 9876543210?

Answer: a) yes; b) no.

Solution. a) Denote by $\to$ the step of multiplying by 2 and by $\Rightarrow$ the step of shuffling digits. Let $\to^k$ denote k successive steps $\to$. Notice that $1 \to^9 512 \Rightarrow 125 \to^3 1000$. Analogously $1000 \to^9 512000 \Rightarrow 125000 \to^3 1000000$, and $1000000 \to^9 512000000 \Rightarrow 125000000 \to^3 1000000000$.

b) When a number not divisible by 3 is multiplied by 2, it remains not divisible by 3. The same is true for shuffling digits, because a number is divisible by 3 iff the sum of its digits (which is preserved by shuffling) is divisible by 3. Since the initial number 1 is not divisible by 3, but 9876543210 is, it is not possible to obtain the number 9876543210.

O2. *(Juniors.)* Which of the numbers 2^{2014} and $3^{303} \cdot 4^{404} \cdot 5^{505}$ is greater?

Answer: The second number.

Solution 1. Note that $3^{303} > 2^{303}$, $4^{404} = (2^2)^{404} = 2^{2\cdot 404} = 2^{808}$, and $5^{505} > 4^{505} = (2^2)^{505} = 2^{2\cdot 505} = 2^{1010}$. Therefore $3^{303} \cdot 4^{404} \cdot 5^{505} > 2^{303} \cdot 2^{808} \cdot 2^{1010} = 2^{303+808+1010} = 2^{2121} > 2^{2014}$.

Solution 2. Using $5^{505} > 3^{505}$ and $3^2 > 2^3$, we get $3^{303} \cdot 4^{404} \cdot 5^{505} > 3^{808} \cdot 4^{404} = \left(3^2\right)^{404} \cdot 4^{404} > \left(2^3\right)^{404} \cdot \left(2^2\right)^{404} = 2^{3\cdot 404+2\cdot 404} = 2^{2020} > 2^{2014}$.

O3. *(Juniors.)* Let ABC be an acute triangle. The arcs AB and AC of the circumcircle of the triangle are reflected over the lines AB and AC, respectively. Prove that the two arcs obtained intersect in another point besides A.

Solution. Let the angles of the triangle ABC at vertices A, B, C be α, β, γ, respectively. Let s be the line tangent to the circumcircle of ABC at A, and let X and Y be points on s so that BX and CY are perpendicular to s (Fig. 1). Then $\angle XAB = \gamma$ and $\angle YAC = \beta$. Let X' be the reflection of X over AB and Y' the reflection of Y over AC. Then the lines AX' and AY' are tangent to the reflections of the arcs AB and AC respectively. Therefore, the reflected arcs intersect iff $\angle BAC < \angle BAX' + \angle CAY'$. Since $\angle BAX' = \angle BAX = \gamma$ and

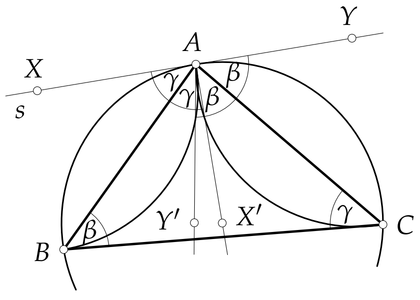

Fig. 1

$\angle CAY' = \angle CAY = \beta$, this is equivalent to $\alpha < \beta + \gamma$. But this holds because, as the triangle is acute, $\alpha < 90^\circ < 180^\circ - \alpha = \beta + \gamma$.

O4. *(Juniors.)* The physical education teacher ordered the 13 girls in the class to form a line such that no girl would have girls shorter than her standing next to her on both sides. All girls in the class are of different height. In how many ways can the girls carry out the teacher's command?

Answer: 4096.

Solution 1. Let s_n be the number of line-ups satisfying the teacher's condition in the case of n girls. Clearly $s_1 = 1$. To line up $n+1$ girls, one can first line up all girls except the tallest one — there are s_n possibilities for that — and then add the tallest girl. Due to the teacher's condition, the tallest girl can never stand between two other girls. She can, however, always stand at either end of the line; since she is taller than her neighbour, the condition will not be broken for her neighbour either. As the tallest girl has to choose between two ends of the line, $s_{n+1} = 2s_n$. This gives $s_{13} = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{12 \text{ factors}} \cdot 1 = 2^{12} = 4096$.

Solution 2. The teacher's condition implies that girls to the left of the shortest girl must stand in increasing order of their height, and the same holds for girls to the right of the shortest girl. To completely determine the line-up, it therefore suffices for each girl, except for the shortest, to choose whether she will stand to the left or to the right of the shortest girl. Since there are 12 girls who will make this choice, there are $2^{12} = 4096$ possible arrangements.

O5. *(Juniors.)* Every cell in an $m \times n$ grid is coloured black or white in such a way that no rectangle formed by the cells with both sides longer than 1 would have four corner cells of the same colour. Find all possible values of n if: a) $m = 2$; b) $m = 3$; c) $m = 4$.

Answer: a) Any n; b) $n \leqslant 6$; c) $n \leqslant 6$.

Solution. a) Colour the grid like a chessboard (Fig. 2). The only rectangles that have to be considered are of size $2 \times k$ ($k \geqslant 2$), and those clearly don't have all corners in the same colour.

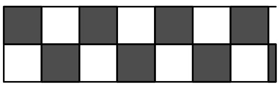

Fig. 2

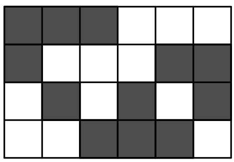

Fig. 3

b), c) A suitable colouring of an $m \times n$ grid gives a suitable colouring of any $m' \times n'$ subgrid, where $m' \leqslant m$ and $n' \leqslant n$. So it remains to show that a suitable colouring does not exist for the 3×7 grid, but does exist for the 4×6 grid. Assume that the 3×7 grid has 3 horizontal rows of 7 cells each. For any colouring, any row has at least 4 cells of the same colour. Without loss of generality, assume that there are 4 black cells in the top row. In either of the bottom two rows, there can be only one black square in those four columns; otherwise a rectangle with four black corner cells would be formed. Thus at least two of those four columns must have white cells in both bottom rows, forming a rectangle with four white corners. A suitable colouring of the 4×6 grid is shown on Fig. 3.

O6. *(Seniors.)* Find all positive integers n for which the equation $(x^2+y^2)^n = (xy)^{2014}$ has positive integer solutions.

Answer: 1908, 1976, 2012.

Solution. Assume that $(x^2 + y^2)^n = (xy)^{2014}$ holds for some positive integers n, x, y. From $x^2 + y^2 \geqslant 2xy > xy$ follows $n < 2014$. Let $d = \gcd(x, y)$ and $a = \frac{x}{d}$, $b = \frac{y}{d}$. Then $d^{2n}(a^2 + b^2)^n = d^{2\cdot 2014}(ab)^{2014}$, which gives the equality $(a^2 + b^2)^n = d^{2\cdot(2014-n)}(ab)^{2014}$. As b divides the right side of this equality, $(a^2 + b^2)^n$ is divisible by b as well. But because $\gcd(a, b) = 1$, also $\gcd(a^2, b) = 1$ and $\gcd(a^2 + b^2, b) = 1$, whence $\gcd((a^2 + b^2)^n, b) = 1$. So the only possibility is $b = 1$. Due to symmetry, also $a = 1$. The above equality now takes the form $2^n = d^{2\cdot(2014-n)}$. Therefore $d = 2^k$ for some k and $n = 2k \cdot (2014 - n)$, whence $n \cdot (2k + 1) = 4k \cdot 1007$. Since $\gcd(2k + 1, 4) = 1$ and $\gcd(2k + 1, k) = 1$, $2k + 1$ divides 1007. As $1007 = 19 \cdot 53$ and n has to be positive, the possible values for k are 9, 26 and 503, and the corresponding values of n are 1908, 1976 and 2012.

O7. *(Seniors.)* Find all real-valued functions that are defined on the set of real numbers and satisfy the equation $f(x + y)f(y) = f(x + xf(y))$ for all real numbers x and y.

Answer: $f(x) = 0$ and $f(x) = 1$.

Solution 1. Taking $x = y = 0$ in the equation gives $f(0)^2 = f(0)$. Therefore, $f(0) = 0$ or $f(0) = 1$. Taking $x = 0$, we get that for every y, $f(y)^2 = f(0)$. If $f(0) = 0$, this implies $f(y) = 0$ for every real number y. If $f(0) = 1$, then for every y either $f(y) = 1$ or $f(y) = -1$. Suppose that for some real number a, $f(a) = -1$. Taking $x = -a$ and $y = a$ in the initial equation gives $f(0)f(a) = f(0)$, whence $f(a) = 1$, a contradiction. Therefore, $f(y) = 1$ for every real number y. It is easy to verify that both of these solutions satisfy the equation.

Solution 2. If there is a real number y for which $f(y) = 0$, substituting it into the equation gives $0 = f(x)$ for every x. If $f(y) \neq 0$ for every real number y, then taking $x = \frac{y}{f(y)}$ in the equation gives $f\big(\frac{y}{f(y)} + y\big) \cdot f(y) = f\big(\frac{y}{f(y)} + \frac{y}{f(y)} \cdot f(y)\big) = f\big(\frac{y}{f(y)} + y\big)$. Dividing both sides by $f\big(\frac{y}{f(y)} + y\big)$, we get that $f(y) = 1$ for every y. Both of these solutions satisfy the equation.

O8. *(Seniors.)* Let ABC be a triangle. Let K, L and M be points on the sides BC, AC and AB, respectively, such that $\frac{|AM|}{|MB|} \cdot \frac{|BK|}{|KC|} \cdot \frac{|CL|}{|LA|} = 1$. Prove that it is possible to choose two triangles out of ALM, BMK, CKL whose inradii sum up to at least the inradius of triangle ABC.

Solution. Denote by r_Δ the inradius of triangle Δ. The condition $\frac{|AM|}{|MB|} \cdot \frac{|BK|}{|KC|} \cdot \frac{|CL|}{|LA|} = 1$ of the problem enables us to assume w.l.o.g. that $\frac{|AM|}{|MB|} \leqslant 1$ and $\frac{|CL|}{|LA|} \geqslant 1$, which are equivalent to $\frac{|CL|}{|LA|} \geqslant \frac{|CK|}{|KB|}$ and $\frac{|BK|}{|KC|} \leqslant \frac{|BM|}{|MA|}$, respectively. Let X and Y be points on sides BA and CA such that $\frac{|BX|}{|XA|} = \frac{|BK|}{|KC|}$ and $\frac{|CY|}{|YA|} = \frac{|CK|}{|KB|}$; then both triangles XBK and YKC are similar to triangle ABC (Fig. 4).

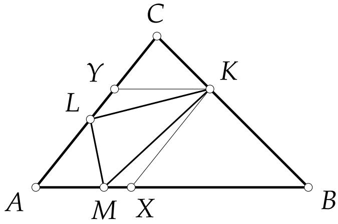

Fig. 4

As the above inequalities show that points Y and X lie on line segments CL and BM, respectively, we obtain $r_{BMK} \geqslant r_{BXK}$ and $r_{CKL} \geqslant r_{CKY}$. Thus $r_{BMK} + r_{CKL} \geqslant r_{BXK} + r_{CKY} = \frac{|BK|}{|BC|} \cdot r_{ABC} + \frac{|CK|}{|CB|} \cdot r_{ABC} = \frac{|BK|+|KC|}{|BC|} \cdot r_{ABC} = \frac{|BC|}{|BC|} \cdot r_{ABC} = r_{ABC}$, which proves the claim.

Remark. This problem, proposed by Estonia, appeared in the IMO 2014 Shortlist as G2.

O9. *(Seniors.)* Consider a grid with n rows and m columns, for positive integers m and n. Find all pairs (k, l) of positive integers for which it is possible to mark some cells of the grid so that there are exactly k marked cells in each row and exactly l marked cells in each column.

Answer: $\left(i \cdot \frac{m}{d}, i \cdot \frac{n}{d}\right)$, where $d = \gcd(n, m)$ and $0 \leqslant i \leqslant d$.

Solution. Let $d = \gcd(n, m)$. We first show that for every suitable pair (k, l), there hold $k = i \cdot \frac{m}{d}$ and $l = i \cdot \frac{n}{d}$, where $0 \leqslant i \leqslant d$. Counting marked cells first by rows and then by columns, we get $nk = ml$, from which $\frac{n}{d} \cdot k = \frac{m}{d} \cdot l$. As the numbers $\frac{n}{d}$ and $\frac{m}{d}$ are coprime, l must be a multiple of $\frac{n}{d}$ and k be a multiple of $\frac{m}{d}$. The last equality shows that the corresponding multipliers are equal. So $k = i \cdot \frac{m}{d}$ and $l = i \cdot \frac{n}{d}$ for some positive integer i. Also $i \leqslant d$, because $i \cdot \frac{m}{d} = k \leqslant m = d \cdot \frac{m}{d}$ (the number of marked cells in a row cannot exceed the number of all cells in a row).

It remains to show that all pairs with the mentioned property actually work. Let i be any integer from interval $[0, d]$. Divide the $n \times m$ grid into $d \times d$ squares. If the cells in every such square are marked in such a way that exactly i cells are marked in each row and in each column of the square, the whole $n \times m$ grid has $i \cdot \frac{m}{d}$ marked cells in each row and $i \cdot \frac{n}{d}$ marked cells in each column, as needed. The $d \times d$ square can be marked in a suitable way by marking i consecutive cells in the first row; and for each following row, cyclically shifting the markings of rows right by one cell.

O10. *(Seniors.)* A barrel of pickles stands next to the fence separating the yards of two neighbours Jaan Tatikas and Salomon Vesipruul. Each neighbour wishes to obtain the barrel for himself. To accomplish that, either neighbour can make the following moves.

- He can go out from his house to either side of the fence.
- If he is at the fence on the same side as the barrel, he can take the barrel (even if the other neighbour is holding it).
- He can climb over the fence. If he is holding the barrel, he takes it along.
- If he is in his own yard and is holding the barrel, he can take it into his house.

The neighbour who manages to take the barrel into his house wins. In the beginning, both neighbours are in their houses. The moves are made alternatingly, with Tatikas making the first move. However, both neighbours are allowed to break the rules at most once by making two moves in succession. Can any neighbour guarantee the victory if the barrel is initially: a) on Tatikas' side; b) on Vesipruul's side?

Answer: a) yes; b) no.

Solution. a) We show that Tatikas can always get the barrel if the barrel is initially in his yard. For his first move, he will go to his side of the fence. Since Vesipruul has no way of taking the barrel to his side even with a double move, Tatikas can next make a double move of taking the barrel and going home.

b) Suppose that the barrel is initially in Vesipruul's yard. If Tatikas remains on his side of the yard after his first move, Vesipruul will be able to win by using a strategy symmetric to the one described in the solution of part a). If Tatikas begins with the double move of going to Vesipruul's side of the fence and taking the barrel, Vesipruul can also win by going to Tatikas's side of the fence. When Tatikas next crosses the fence with the barrel, Vesipruul will make a double move, taking the barrel and climbing to his side. Since the double move has been used, Tatikas has no way of stopping Vesipruul from taking the barrel home on his next turn.

So as not to lose immediately, Tatikas must begin by going to Vesipruul's side of the fence, but not taking the barrel yet. If Vesipruul would now make a double move, going to his side of the fence and taking the barrel, Tatikas could win by taking the barrel from him and crossing the fence. If Vesipruul would instead go to Tatikas's side on his first move, Tatikas could win by taking the barrel, as follows. Since he is threatening to make a double move of crossing the fence and going to his house, Vesipruul will be forced to cross the fence to his side and take the barrel from Tatikas. But then Tatikas can make a double move, retaking the barrel and crossing the fence, which, as seen before, is a victory condition for him. Vesipruul's only remaining first move is going to his side of the fence without taking the barrel. As Vesipruul could next take the barrel and win with a double move, Tatikas's only chance is to make a double move himself, taking the barrel and crossing the fence to his side. Now, Vesipruul's only way to avoid losing is making a double move by crossing the fence and taking the barrel. Next, both neighbours can prevent the other from winning by grabbing the barrel on each move; so neither will be able to guarantee a victory.

O11. *(Seniors.)* Does there exist an integer x for which $2 \leqslant x \leqslant m-1$ and $x^2 - x$ is divisible by m, if a) $m = 2014$; b) $m = 2015$?

Answer: a) yes; b) yes.

Solution. a) Factorising gives $2014 = 2 \cdot 19 \cdot 53$. Taking $x = 19 \cdot 53 = 1007$, we get $x - 1 = 1006 = 2 \cdot 503$, i.e., $x^2 - x = x(x-1) = 2014 \cdot 503$.

b) Factorising gives $2015 = 5 \cdot 13 \cdot 31$. Taking $x = 2 \cdot 13 \cdot 31 = 806$, we get $x - 1 = 805 = 5 \cdot 161$, i.e., $x^2 - x = x(x-1) = 2015 \cdot 2 \cdot 161$.

Remark: these are not the only possibilities. All suitable values for x are 266, 742, 1007, 1008, 1273 and 1749 in case a), and 156, 651, 806, 1210, 1365 and 1860 in case b).

O12. *(Seniors.)* Find all real solutions of the system of equations $x(y-1) + y(x+1) = 6$, $(x-1)(y+1) = 1$.

Answer: $x = \frac{4}{3}, y = 2$ or $x = -2, y = -\frac{4}{3}$.

Solution 1. Opening brackets and combining constant terms in both equations gives $2xy - x + y = 6$, $xy + x - y = 2$. Subtracting from first equation the second equation multiplied by 2, we get $3y - 3x = 2$, whereas summing those equations gives $3xy = 8$. The first of these equalities gives $3y = 3x + 2$. Substituting this into the second equality gives $x(3x+2) = 8$ or $3x^2 + 2x - 8 = 0$. The solutions of this quadratic equation are $x_{1,2} = \frac{-2 \pm \sqrt{2^2 + 4 \cdot 3 \cdot 8}}{6} = \frac{-2 \pm 10}{6}$, i.e., $x_1 = \frac{4}{3}$ and $x_2 = -2$, and consequently $y_1 = 2$ and $y_2 = -\frac{4}{3}$. We can verify that both solutions satisfy the original system.

Solution 2. From the second equation we get $y + 1 = \frac{1}{x-1}$, whence $y = \frac{1}{x-1} - 1 = \frac{2-x}{x-1}$ and $y - 1 = \frac{1}{x-1} - 2 = \frac{3-2x}{x-1}$. Substituting these into the first equation gives $\frac{x(3-2x)}{x-1} + \frac{(2-x)(x+1)}{x-1} = 6$ or $x(3-2x) + (2-x)(x+1) = 6(x-1)$. Opening brackets and combining like terms yields the quadratic equation $-3x^2 - 2x + 8 = 0$ whose solutions are $x_1 = \frac{4}{3}$ and $x_2 = -2$, and consequently $y_1 = 2$ and $y_2 = -\frac{4}{3}$.

O13. *(Seniors.)* Find all four-digit natural number whose ratio with its sum of digits is the least possible.

Answer: 1099 is the only such number.

Solution. If the sum of digits is fixed, then the number itself, and consequently the ratio under consideration, is the smallest possible if the initial digits are as small as possible and the final digits are as large as possible. Hence it suffices to compare the ratios in the case of four-digit numbers that can be obtained from the number 1000 by stepwise increasing its digits up to 9, starting from the last digit (i.e., in the case of numbers 1000 through 1009, 1019 through 1099, 1199 through 1999 and 2999 through 9999).

We show that, among these numbers, the ratio of the number and its sum of digits is the smallest in the case of 1099. Indeed, when increasing its second and then first digit by one at a time, the number increases at least $\frac{1999}{1899} > \frac{20}{19}$ times each step, while the sum of its digits increases at most $\frac{20}{19}$ times each step. Hence the ratio increases. Likewise, when starting from 1099 and decreasing the third and then the fourth digit by one at a time, the number decreases at most $\frac{1019}{1009} < \frac{101}{100}$ times, while the sum of digits decreases at least $\frac{19}{18} > \frac{101}{100}$ times. Hence the ratio increases here, too.

O14. *(Seniors.)* Find all real-valued functions f that are defined on the set of real numbers and satisfy the condition $f(x^2) + f(xy) = f\big(f(x+y)\big)$ for any real numbers x and y.

Answer: only the constant function $f(x) = 0$.

Solution 1. Taking $x = 0$ and $y = a$ in the given equation, we get $2f(0) = f(f(a))$ which must hold for any real number a. Taking $a = x + y$, we get $f(x^2) + f(xy) = 2f(0)$. Now taking $x = 1$ gives $f(y) = 2f(0) - f(1)$, where the right side is constant. Hence f is a constant function: $f(x) = c$, with c some fixed real number. Substituting it into the initial equation, we get $c + c = c$, whence $c = 0$. Therefore the only possible solution is $f(x) = 0$. It can be verified to satisfy the initial equation.

Solution 2. Since $x + y = y + x$, for any real numbers x and y $f(x^2) + f(xy) = f(f(x+y)) = f(f(y+x)) = f(y^2) + f(yx)$, therefore $f(x^2) = f(y^2)$. So for any non-negative a, $f(a) = f\big((\sqrt{a})^2\big) = f(0)$, i.e., f is constant for non-negative arguments. Now taking $x = -y$, we get $f(x^2) + f(-x^2) = f(f(0))$ and thus $f(-x^2) = f(f(0)) - f(0)$, meaning f is also constant for all non-positive arguments. Since the number 0 is both non-negative and non-positive, the constants $f(0)$ and $f(f(0)) - f(0)$ are equal and f is constant on the whole real line. We now find the only fitting function f as in solution 1.

Solution 3. Taking $x = 0$ in the equation, we see that $2f(0) = f(f(y))$ for any real number y. Taking $y = 0$ in the initial equation and taking into account that $f(f(x)) = 2f(0)$, we get $f(x^2) + f(0) = f(f(x)) = 2f(0)$, i.e., $f(x^2) = f(0)$ for any x. Now taking $y = -x$ in the initial equation, we see that for any real number x, $f(x^2) + f(-x^2) = f(f(0))$, from which $f(0) + f(-x^2) = 2f(0)$, i.e., $f(-x^2) = f(0)$. Since any real number a can be expressed as either $a = x^2$ or $a = -x^2$, the function f must be constant. We now find the only fitting function f as in solution 1.

Solution 4. Taking $x = 0$ gives $2f(0) = f(f(y))$ for any real number y. Taking $y = x$ in the initial equation and taking into account that $f(f(2x)) = 2f(0)$, we get $2f(x^2) = f(f(2x)) = 2f(0)$, i.e., $f(x^2) = f(0)$ for any x. Now, taking $y = 1$ in the initial equation gives $f(x^2) + f(x) = f(f(x+1))$, which is equivalent to $f(x) = f(f(x+1)) - f(x^2) = 2f(0) - f(0) = f(0)$, i.e., f is a constant function. We now find the only fitting function f as in solution 1.

O15. *(Seniors.)* The triangle K_2 has as its vertices the feet of the altitudes of a non-right triangle K_1. Find all possibilities for the sizes of the angles of K_1 for which the triangles K_1 and K_2 are similar.

Answer: $60°, 60°, 60°$ or $\frac{180°}{7}, \frac{360°}{7}, \frac{720°}{7}$.

Solution. Let the vertices of K_1 be A, B and C, the respective angles α, β and γ, and the feet of the altitudes from those vertices A', B' and C' respectively. Consider two cases.

1) Let K_1 be acute. As $\angle AA'B = \angle BB'A = 90°$, $ABA'B'$ is a cyclic quadrilateral (Fig. 5). Thus $\angle B'A'A = \angle ABB' = 90° - \alpha$. Analogously, from the cyclic quadrilateral $ACA'C'$, $\angle C'A'A = 90° - \alpha$. There the angle of K_2 at vertex A' is $180° - 2\alpha$. Similarly, the angles at vertices B' and C' are $180° - 2\beta$ and $180° - 2\gamma$ respectively. Therefore, the triangles K_1 and K_2 are similar if $(180° - 2\alpha,\ 180° - 2\beta,\ 180° - 2\gamma)$ is a rearrangement of (α, β, γ). Without loss of generality, let $\alpha \leqslant \beta \leqslant \gamma$, then $180° - 2\gamma \leqslant 180° - 2\beta \leqslant 180° - 2\alpha$ and therefore $\alpha = 180° - 2\gamma$, $\beta = 180° - 2\beta$ and $\gamma = 180° - 2\alpha$. Now we obtain $3\beta = 180°$ and $\alpha = 4\alpha - 180°$, from which $\alpha = \beta = 60°$ and therefore also $\gamma = 60°$.

2) Let K_1 be obtuse. Without loss of generality, let γ be the obtuse angle. Computing angles like in the first case, we find that the angles of K_2 are 2α, 2β and $2\gamma - 180°$ (Fig. 6). Therefore $(2\alpha, 2\beta, 2\gamma - 180°)$ must be a rearrangement of (α, β, γ). Since no angle can equal itself multiplied by two and the pair (α, β) cannot equal the pair $(2\beta, 2\alpha)$, we can assume without loss of generality that $\alpha = 2\gamma - 180°$, $\beta = 2\alpha$ and $\gamma = 2\beta$. From this we get $\alpha = 8\alpha - 180°$, whence $\alpha = \frac{180°}{7}$, $\beta = \frac{360°}{7}$ and $\gamma = \frac{720°}{7}$.

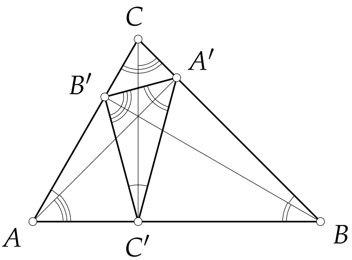

Fig. 5

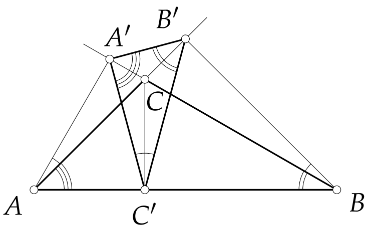

Fig. 6

O16. *(Seniors.)* The hoots of Uhhuu tribe consist of letters U and H only; no spaces or punctuation marks are used. The tribe follows a taboo according to which a non-empty initial part of any hoot may never coincide with the immediately following part of the same length in the same hoot. For example, hoots UUHUU and HUHUU are prohibited while UHHUH and HUHHU are allowed. The chieftain of the tribe checks validity of each new hoot by comparing the initial parts with lengths $1, 2, \ldots$ with the immediately following parts of the same length letter-by-letter from left to right until the first difference appears. Of course he checks the initial parts only whose length does not exceed half of the length of the whole hoot.

Let $l(H)$ be the number of letters in hoot H and let $c(H)$ be the total number of letter comparisons the chieftain makes when checking hoot H. Is is possible to construct a valid hoot H for which $\frac{c(H)}{l(H)} > 100$?

Answer: yes.

Solution. For any hoot H, let $\overline{H}$ denote the hoot obtained from H by changing the last letter. Define hoots H_k, $k = 0, 1, 2, \ldots$ by $H_0 =$ H, $H_{k+1} = H_k\overline{H_k}$ (i.e., $H_0 =$ H, $H_1 =$ HU, $H_2 =$ HUHH etc.). Obviously $l(H_k) = 2^k$.

Let us prove that all H_k are admissible. Let i be the length of the initial part of the hoot currently under checking. Then $0 < i \leqslant 2^{k-1}$. Let $i = p \cdot 2^j$, where p is odd. If $p = 1$ then the initial part of H_k with length i equals H_j and the part immediately following it having the same length equals $\overline{H_j}$, hence they are not the same. Now let $p > 1$. If we divide hoot H_k into parts of length 2^{j+1}, then in each part all letters except the last one are the same. In particular, the 2^jth letter of H_k repeats with period 2^{j+1}. Since $p + 2$ is odd, the letter at position $(p + 2) \cdot 2^j$ coincides with the letter at position 2^j, being different from the letter at position 2^{j+1}. But the letter at position $(p + 2) \cdot 2^j$ in hoot H_k equals the letter at position 2^{j+1} in the part of H_k immediately following the first i letters. Hence the initial part consisting of i letters is not the same as the part immediately following it.

To estimate $c(H_k)$, let v_i be the number of letter comparisons in checking the initial part consisting of i letters. If $i = p \cdot 2^j$ with p odd then the first $2^j - 1$ letters coincide with the letters immediately following the part of length i. For $0 \leqslant j < k - 1$, the numbers 1, 2, 3, …, 2^{k-1} contain 2^{k-2-j} numbers of the form $p \cdot 2^j$ with p odd; in addition, there is 2^{k-1}. Hence $c(H_k) = v_1 + v_2 + v_3 + \ldots + v_{2^{k-1}} \geqslant 2^{k-2} \cdot 1 + 2^{k-3} \cdot 2 + 2^{k-4} \cdot 4 + \ldots + 2 \cdot 2^{k-3} + 1 \cdot 2^{k-2} + 2^{k-1} = 2^{k-2} \cdot (k-1) + 2^{k-1} = 2^{k-2} \cdot (k+1)$. Consequently $\frac{c(H_k)}{l(H_k)} \geqslant \frac{2^{k-2} \cdot (k+1)}{2^k} = \frac{k+1}{4}$. In particular $\frac{c(H_{400})}{l(H_{400})} > 100$.

Selected Problems from the Final Round of National Olympiad

F1. *(Grade 9.)* Does there exist a natural number such that dropping the first three digits from it leads to exactly k times smaller number, where: a) $k = 2015$; b) $k = 2016$?

Answer: a) no; b) yes.

Solution. Let a be the number formed by the three leading digits of given number and let b be the remaining part. Let m be the total number of digits.

a) From the equation $a \cdot 10^{m-3} + b = 2015b$, we obtain $a = \frac{2014b}{10^{m-3}}$. As $\gcd(2014, 10^{m-3}) = 2$, reducing the fraction leaves factor 1007 in the numerator. Thus a cannot be a 3-digit number.

b) For example, 4032 satisfies all conditions since $4032 = 2016 \cdot 2$.

Remark. It is not hard to show that the numbers satisfying the conditions in part b) are precisely those in the form $4032 \cdot 10^t$ and $8064 \cdot 10^t$, where t is an arbitrary non-negative integer.

F2. *(Grade 9.)* The size of the angle at vertex A of parallelogram $ABCD$ is $60°$. The bisectors of angles at vertices B and D intersect sides AD and BC, respectively, at points E and F, respectively. Given that $BEDF$ is a rhombus, find the ratio of the lengths of the parts of line segment BE into which line segment AC divides it.

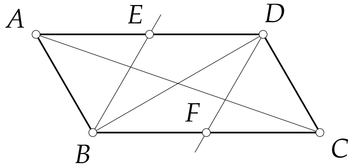

Fig. 7

Answer: 2 : 1.

Solution 1. The angle sizes of parallelogram $ABCD$ are $60°$ and $120°$ as given. Since BE bisects the angle at vertex B, we have $\angle ABE = 60° = \angle BAE$ (Fig. 7). Thus $|AE| = |BE|$. On the other hand, $BEDF$ being a rhombus implies $|BE| = |DE|$. Consequently, $|AE| = |DE|$. Thus BE is the median of triangle ABD drawn from vertex B. As the diagonals of parallelogram bisect each other, the median of ABD drawn from vertex A must lie entirely on line AC. Medians, however, cut each other into parts with length ratio 2 : 1.

Solution 2. The angle sizes of parallelogram $ABCD$ are $60°$ and $120°$ as given. As DF bisects the angle at vertex D, we have $\angle CDF = 60° = \angle DCF$. Hence $|CF| = |DF|$. On the other hand, $BEDF$ being a rhombus implies $|DF| = |BF|$. Consequently, $|CF| = |BF|$. Also $|BC| = |AD|$ and $|BF| = |DE|$ together imply $|FC| = |AE|$. Let AC and BE intersect at N; triangles BCN and EAN are similar because of parallel sides. As $|BC| = 2|FC| = 2|AE|$, their similarity ratio is 2. Consequently, $|BN| : |NE| = 2 : 1$.

F3. *(Grade 9.)* Siim has a dice with faces containing numbers from 1 to 6, and a squared paper whose squares equal by size to the faces of the dice. Siim paints one square red and places the dice on the paper with the face containing number 1 down and covering the red square entirely. Siim starts rolling the dice step-by-step from face to face in such a way that, after each step, the downwards face of the dice covers an entire square of the paper again. Find the smallest number of steps (i.e., rollings of the dice from a face to its neighbouring face over their common edge) that Siim must complete in order to have the dice laying on the red square at least once with each of its faces down and finally returning to the original placement on the red square with the face containing number 1 down.

Answer: 24.

Solution. By conditions of the problem, the dice must leave from the red square and return there at least 6 times with different face downwards. Each such "trip" requires at least 4 steps, as the number of steps during the "trip" is even (colouring the squares like on chessboard we see that the colour of the square containing the dice changes at each step) and with 2 steps, one can only roll the dice away and back onto the same face.

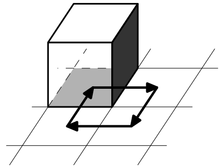

Fig. 8

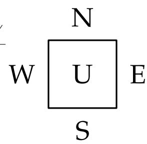

Fig. 9

Next we show that 4 steps is enough for interchanging the downwards face to whatever neighbouring face. For this, roll the dice on the first step so that the face to be moved downwards moved actually downwards, and make each of the 3 following steps in 90° clockwise direction w.r.t. the previous step. This way, the dice lies on the same face after the fourth step as after the first step (in Fig. 8, this face is coloured dark).

It remains to notice that, by choosing a neighbouring face each time, it is possible to go through all faces of the dice and return to the face one started with. Indeed, let the downwards face be D, upwards face be U, and the other faces denoted according to cardinal directions, a suitable cycle is E-U-S-W-D (Fig. 9 depicts the view from above).

F4. *(Grade 9.)* Jüri discovers in the library that a book has some number of consecutive pages torn out. Find the page numbers of the missing pages, given that their sum is 452 and the print is double-sided.

Answer: 53 through 60.

Solution. Let n be the number of missing pages and let their page numbers be $a, a+1, \ldots, a+n-1$. By conditions, $a + (a+1) + \ldots + (a+n-1) = 452$. Thus the arithmetic mean of the page numbers of the missing pages is $\frac{452}{n}$. As consecutive integers lie symmetrically w.r.t. their arithmetic mean, we must have $\frac{452}{n} = \frac{a+(a+n-1)}{2}$, implying $n \cdot (2a + n - 1) = 904 = 2^3 \cdot 113$. As the number n of missing pages is even, $2a + n - 1$ is odd. The odd divisors of 904 are 1 and 113, but the case $2a + n - 1 = 1$ is impossible, as a must be positive. Thus $2a + n - 1 = 113$ and $n = 8$. Solving the equation $2a + 7 = 113$ gives $a = 53$. Therefore, the page numbers of the pages torn out are 53 through 60.

F5. *(Grade 10.)* Find all natural numbers n greater than 1 such that both $\frac{1}{n}$ and $\frac{1}{n+1}$ can be written in the form of finite decimal fraction.

Answer: 4.

Solution. Let $\frac{1}{n}$ and $\frac{1}{n+1}$ be finite decimal fractions where the numbers of digits after the decimal point are u and v, respectively. Then $\frac{10^u}{n}$ and $\frac{10^v}{n+1}$ are integers, meaning that neither n nor $n+1$ has prime divisors other than 2 and 5. As consecutive integers cannot be both divisible by 2 or both divisible by 5, one of them must be a power of 2 and one — a power of 5. As only 1 is a power of both 2 and 5, but we are looking for numbers greater than 1, we have $5^k \pm 1 = 2^l$ where k and l are positive integers. As $5 \equiv 1 \pmod 4$, we have $5^k \equiv 1 \pmod 4$, whence $5^k + 1$ is divisible by 2 but not by 4. Such a number could be a power of 2 only if equal to 2, leading to $5^k = 1$. Consequently, $5^k - 1 = 2^l$. Then $2^l = (5-1) \cdot (5^{k-1} + \ldots + 5 + 1)$, implying that also $5^{k-1} + \ldots + 5 + 1$ must be a power of 2. One possibility is $k = 1$, i.e., $5^{k-1} + \ldots + 5 + 1 = 1$ implying $n = 4$. In the other case, $k > 1$, the number k must be even since the sum $5^{k-1} + \ldots + 5 + 1$ contains k odd summands. This enables the factorization $2^l = 5^k - 1 = \left(5^{\frac{k}{2}} - 1\right) \cdot \left(5^{\frac{k}{2}} + 1\right)$, where both factors must still be powers of 2. As these factors differ by 2, the only possibility is $5^{\frac{k}{2}} - 1 = 2$ and $5^{\frac{k}{2}} + 1 = 4$, leading to contradiction.

F6. *(Grade 10.)* Let x, y, z be real numbers such that $x + y + z = 1$, $x^2 + y^2 + z^2 = 11$ and $xyz = 3$. Find $x^3 + y^3 + z^3$.

Answer: 25.

Solution. The first equation cubed is

$$x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 3y^2z + 3xz^2 + 3yz^2 + 6xyz = 1.$$

Multiplying the first two equations gives

$$x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 = 11.$$

Subtracting from the first equation the second equation multiplied by 3 leads to

$$-2 \cdot (x^3 + y^3 + z^3) + 6 \cdot xyz = -32.$$

Since $xyz = 3$, we have $-2 \cdot (x^3 + y^3 + z^3) = -32 - 6 \cdot 3 = -50$, implying $x^3 + y^3 + z^3 = 25$.

F7. *(Grade 10.)* Point P on side BC of triangle ABC satisfies $|BP| : |PC| = 2 : 1$. Prove that line AP bisects the median of triangle ABC drawn from vertex C.

Solution 1. Let Q be the midpoint of line segment BP; the conditions of the problem imply $|BQ| = |QP| = |PC| = \frac{1}{3}|BC|$. Let R be the midpoint of line segment AB (Fig. 10). Then RQ is a midline of ABP. Consequently,

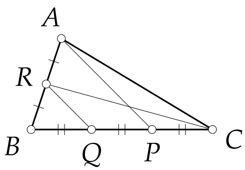

Fig. 10

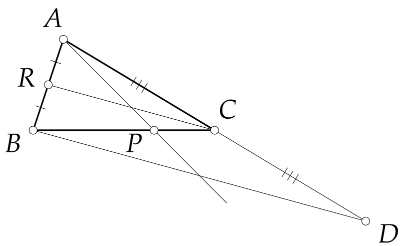

Fig. 11

$RQ \parallel AP$. Ray AP bisects side CQ of triangle CRQ while being parallel to side RQ of this triangle. Thus AP extends the midline of triangle CRQ and bisects therefore also its side CR. But line segment CR is the median of triangle ABC drawn from vertex C.

Solution 2. Let R be the midpoint of segment AB. Choose point D on ray AC beyond point C such that $|AC| = |CD|$ (Fig. 11). Then BC is a median of triangle ABD. As $|BP| : |PC| = 2 : 1$, point P is the intersection point of medians of triangle ABD. Thus AP lies entirely on the other median of triangle ABD, i.e., ray AP bisects the segment BD. As CR is the midline of triangle ABD, we have $CR \parallel BD$, implying that ray AP also bisects the segment CR. But this is the median of triangle ABC drawn from vertex C.

F8. *(Grade 10.)* On a blackboard, natural numbers 1 through n are written in one row and the same numbers in reversed order in the second row below the first. Denote by $S(n)$ the sum of products of all pairs of numbers on top of each other. (For example, $S(6) = 1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1 = 56$.) Prove that for every natural number n, $S(n+1) - S(n) = 1 + 2 + \ldots + n + (n+1)$ and $S(n+1) + S(n) = 1^2 + 2^2 + \ldots + n^2 + (n+1)^2$.

Solution. Write the terms in both $S(n+1)$ and $S(n)$ under each other:

$$\begin{array}{lclclclclclcl} S(n+1) & = & 1\cdot(n+1) & + & 2\cdot n & & & + & \ldots & + & n\cdot 2 & + & (n+1)\cdot 1 \\ S(n) & = & 1\cdot n & & & + & 2\cdot(n-1) & + & \ldots & + & n\cdot 1 & & \end{array}$$

By subtracting the corresponding terms we come to $S(n+1) - S(n) = 1 \cdot ((n+1) - n) + 2 \cdot (n - (n-1)) + \ldots + n \cdot (2-1) + (n+1)$. As all differences are equal to 1, we get $S(n+1) - S(n) = 1 + 2 + \ldots + n + (n+1)$. On the other hand, adding corresponding terms gives $S(n+1) + S(n) = 1 \cdot ((n+1) + n) + 2 \cdot (n + (n-1)) + \ldots + n \cdot (2+1) + (n+1)$. The right-hand factors are odd numbers $2n+1, 2n-1, \ldots, 3, 1$ which can be rewritten as $(n+1)^2 - n^2$, $n^2 - (n-1)^2$, $\ldots$, $2^2 - 1^2$, 1^2. Hence $S(n+1) + S(n) = \big((n+1)^2 - n^2\big) + \big(2n^2 - 2(n-1)^2\big) + \ldots + \big(n \cdot 2^2 - n \cdot 1^2\big) + (n+1) \cdot 1^2$. In each "big parentheses", second term cancels with the first term in the next parentheses, leaving the latter with coefficient 1. After all cancellations, rearrangement of terms gives $S(n+1) + S(n) = 1^2 + 2^2 + \ldots + n^2 + (n+1)^2$.

F9. *(Grade 10.)* Does there exist a right triangle with integral side lengths whose one side length equals another side length squared?

Answer: no.

Solution. Suppose that two side lengths of a right triangle with integral side lengths are a and a^2. As $a^2 \geqslant a$, we have two possibilities: either the side with length a^2 is the hypotenuse of the triangle or the sides with lengths a and a^2 are both legs. Let the third side length be x. If the side with length a^2 is the hypotenuse, Pythagorean theorem implies $a^4 = a^2 + x^2$. Then $x^2 = a^4 - a^2 = a^2(a^2 - 1)$, i.e., x^2 is divisible by a^2 and hence x is divisible by a. Thus $a^2 - 1$ is the square of integer $\frac{x}{a}$. However, no two positive perfect squares differ by 1. If the sides with lengths a and a^2 are legs, Pythagorean theorem implies $x^2 = a^4 + a^2 = a^2(a^2+1)$, i.e., x^2 is divisible by a^2 and x is divisible by a again. Thus a^2+1 is the square of integer $\frac{x}{a}$ which is impossible as a^2+1 differs from a^2 by 1.

Remark. From factorization $x^2 = a^2(a^2-1)$ or $x^2 = a^2(a^2+1)$, contradiction can also be obtained in other ways. One possibility is to use the fact that consecutive integers are relatively prime, but whenever the product of relatively prime number is a perfect square, the factors themselves must be perfect squares. The contradiction is derived from here as in the Solution. Another possibility is to notice that if the product of two positive integers is the square of some integer x, then x must be between these two factors. But there are no integers between two consecutive integers. For example, identity $x^2 = a^4 + a^2$ implies $x^2 < a^4 + 2a^2 + 1 = (a^2+1)^2$ showing that x^2 lies between the squares of consecutive integers a^2 and a^2+1. From identity $x^2 = a^4 - a^2$, we similarly get $(a^2)^2 > x^2 > a^4 - 2a^2 + 1 = (a^2-1)^2$, where the second equality holds because of $a > 1$ (which itself is implied by $a^4 > x^2 \geqslant 1$). Such approach doesn't even need factorization.

F10. *(Grade 11.)* Find all positive integers n which can be represented as a prime power with a positive integral exponent such that the equation $x^3 - 2x^2 + 2x - 1 = n$ has an integral solution.

Answer: 3.

Solution. Let $n = p^k$ where p is a prime number and k is a positive integer. Note that $x^3 - 2x^2 + 2x - 1 = (x-1)(x^2-x+1)$. As the quadratic function $y = x^2 - x + 1$ takes only positive values because of negative discriminant, the equality $(x-1)(x^2-x+1) = p^k$ can hold only if both $x-1$ and x^2-x+1 are powers of p. As $x^2-x+1 = x(x-1)+1$, we have $\gcd(x-1, x^2-x+1) = 1$, implying either $x-1 = 1$ or $x^2-x+1 = 1$. The latter is impossible because $x^2 - x = 0$ would give $x = 0$ or $x = 1$ so $x-1$ would not be positive. Hence $x-1 = 1$, implying $x = 2$ and $(x-1)(x^2-x+1) = 3 = 3^1$.

F11. *(Grade 11.)* a) Tiina has written a positive integer into her exercise book. For math training, she every day writes into the exercise book a new number that equals the product of all digits of the number written there last time. Prove that, starting from some day, she continues writing one and the same number.

b) Would the same claim hold irrespectively of the initial number also in the case when Tiina multiplied the product of all digits of the previous number each time by 2?

Answer: b) no.

Solution. a) We show that the product of any multiple-digit number is less than the number itself. Let us have a number $\overline{d_k d_{k-1} \ldots d_1 d_0}$, where d_i are the digits of the number and $k > 0$. We get $\overline{d_k d_{k-1} \ldots d_1 d_0} = d_k \cdot 10^k + \overline{d_{k-1} \ldots d_1 d_0} \geqslant d_k \cdot 10^k = d_k \cdot (10 \cdot 10 \cdot \ldots \cdot 10) > d_k \cdot d_{k-1} \cdot \ldots \cdot d_1 \cdot d_0$, where the last inequality holds thanks to $d_{k-1}, \ldots, d_1, d_0$ being less than 10. Hence every new number written by Tiina is less than previous ones, until she gets a 1-digit number which repeats forever.

b) Starting from number 4, one gets the cycle $4 \to 8 \to 16 \to 12 \to 4 \to \ldots$ Thus one not necessarily gets just one repeating number.

F12. *(Grade 11.)* How many 10-digit numbers are there which are divisible by 99 and whose all digits are different?

Answer: 285120.

Solution. The sum of digits of every number under consideration is 45. Hence they are all divisible by 9. For being divisible by 11, the difference of the sum of digits at even positions and the sum of digits at odd positions must be divisible by 11. Let s be the sum of digits at even positions. Then $s-(45-s)$ must be divisible by 11, i.e., $11 \mid 2s-45$. This holds if and only if $s \equiv 6 \pmod{11}$. As $0+1+2+3+4 \leqslant s \leqslant 5+6+7+8+9$, the only possibilities are $s = 17$ and $s = 28$. Note that in the latter case, the sum of the remaining digits equals 17. The sum 17 can be obtained from 5 different digits in 11 different ways: $\{0,1,2,5,9\}$, $\{0,1,2,6,8\}$, $\{0,1,3,4,9\}$, $\{0,1,3,5,8\}$, $\{0,1,3,6,7\}$, $\{0,1,4,5,7\}$, $\{0,2,3,4,8\}$, $\{0,2,3,5,7\}$, $\{0,2,4,5,6\}$, $\{1,2,3,4,7\}$, $\{1,2,3,5,6\}$. Each number under consideration must have namely the digits of one of these sets in some order either at even positions or at odd positions. Thus the choice of the number can be made in four steps: 1) choose the position for digit 0, for which there are 9 possibilities; 2) choose arbitrarily one of the sets above, for which there are 11 possibilities (as the position of 0 has been fixed, the choice of the set also determines whether its digits must be placed at even or at odd positions); 3) choose the order of the digits whose position number is of the same parity as digit 0, for which there are $4! = 24$ possibilities; 4) choose the order of the remaining digits at the remaining positions, for which there are $5! = 120$ possibilities. Altogether, there are $9 \cdot 11 \cdot 24 \cdot 120 = 285120$ ways to construct the desired number.

F13. *(Grade 11.)* The area S and the circumradius R of a triangle satisfy $S \geqslant R^2$. Prove that the sizes of all angles of the triangle are greater than 30° while none of them is greater than 90°.

Solution 1. Let α be the size of an arbitrary angle of the triangle. Denote the vertices of the triangle by A, B, C, in such a way that the angle of size α lies at vertex A. Let the lengths of sides opposite to A, B, C be a, b, c, respectively. As the sides of the triangle are chords of its circumcircle, none of the lengths can exceed the diameter of the circumcircle.

For showing the first inequality, use this property for sides of lengths b and c, i.e., $b \leqslant 2R$ and $c \leqslant 2R$. As at most one side can coincide with the diameter of the circumcircle, at least one of these inequalities is strict. Using the assumption of the problem thus gives $R^2 \leqslant S = \frac{1}{2}bc\sin\alpha < \frac{1}{2} \cdot 2R \cdot 2R \cdot \sin\alpha = 2R^2 \cdot \sin\alpha$, implying $2\sin\alpha > 1$. Hence $\alpha > 30^\circ$.

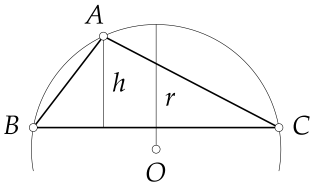

Fig. 12

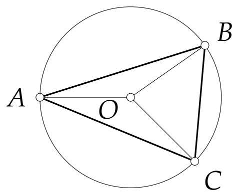

Fig. 13

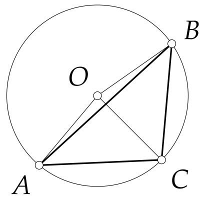

Fig. 14

To prove the second inequality, use the same property for side of length a, i.e., $a \leqslant 2R$. Let h be the altitude from vertex A. By the assumption of the problem, $R^2 \leqslant S = \frac{1}{2}ah \leqslant \frac{1}{2} \cdot 2R \cdot h = R \cdot h$, implying $h \geqslant R$. If the angle of triangle ABC at vertex A were obtuse, the circumcentre O of triangle ABC would lie outside the triangle, implying $h < R$ (Fig. 12). Contradiction shows that the angle at vertex A must be either acute or right, i.e., $\alpha \leqslant 90^\circ$.

Solution 2. Denote the side lengths and angle sizes of the triangle as in Solution 1. By the law of sines, $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$. Hence $S = \frac{1}{2}ab\sin\gamma = \frac{1}{2} \cdot 2R\sin\alpha \cdot 2R\sin\beta \cdot \sin\gamma = 2R^2\sin\alpha\sin\beta\sin\gamma$. Since $S \geqslant R^2$, we obtain $\sin\alpha\sin\beta\sin\gamma \geqslant \frac{1}{2}$. As the sine of an angle cannot exceed 1, all of $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ must be greater than $\frac{1}{2}$ (equality could occur for one angle if the sines of the other angles were 1, but a triangle cannot have two right angles). Hence the sizes of the angles of the triangle are greater than 30°. Suppose that the triangle is obtuse, and let its obtuse angle have size γ. The inequality $\sin\alpha\sin\beta\sin\gamma \geqslant \frac{1}{2}$ then implies $\sin\alpha\sin\beta > \frac{1}{2}$. As $\alpha + \beta < 90^\circ$, we obtain $\sin\alpha\sin\beta < \sin\alpha\sin(90^\circ - \alpha) = \sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha \leqslant \frac{1}{2}$, a contradiction.

Solution 3. Denote the vertices and angle sizes of the triangle as in Solution 1. W.l.o.g., $\alpha \leqslant \beta \leqslant \gamma$. Let O be the circumcentre of the triangle; then the area of the triangle equals the sum of the areas of triangles OBC, OCA and OAB (Fig. 13). Consequently, $S = \frac{1}{2}R^2 \cdot (\sin 2\alpha + \sin 2\beta + \sin 2\gamma)$. (This formula holds for obtuse triangle as well, in the case of which it corresponds to subtraction of the area of triangle OAB from the sum of the areas of triangles OBC and OCA, see Fig. 14.) Since $S \geqslant r^2$, this implies $\sin 2\alpha + \sin 2\beta + \sin 2\gamma \geqslant 2$. If the triangle were obtuse, one of the summands in the l.h.s. would be negative. As neither of the other summands can be greater than 1, this inequality could not be valid.

It remains to show that the sizes of all angles are greater than 30°. Relying on the first part of the solution, we can assume that $2\beta \leqslant 2\gamma \leqslant 180^\circ$. By Jensen's inequality, $\frac{\sin 2\beta + \sin 2\gamma}{2} \leqslant \sin\frac{2\beta + 2\gamma}{2} = \sin(\beta + \gamma) = \sin\alpha$, implying $\sin 2\beta + \sin 2\gamma \leqslant 2\sin\alpha$. Hence $\alpha \leqslant 30^\circ$ would lead to $2\sin\alpha \leqslant 1$ and $\sin 2\alpha \leqslant \frac{\sqrt{3}}{2}$, giving $\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leqslant \frac{\sqrt{3}}{2} + 1 < 2$, which contradicts the inequality above. Consequently, $\alpha > 30^\circ$. As α is the least angle by choice, the other sizes of the angles must be greater than 30° as well.

Remark: Angle size 90° can be obtained (an isosceles right triangle satisfies the conditions of the problem). The least possible angle size is nevertheless greater than 30°, being approximately 32.9°.

F14. *(Grade 12.)* Prove that the equation $x^2 + y^3 = z^{2015}$ has infinitely many positive integral solutions.

Solution 1. Let k be any nonnegative integer and $n = 2015 \cdot (6k+5) - 1 = 6 \cdot (2015k + 1679)$. Then positive integers $x = 2^{\frac{n}{2}}$, $y = 2^{\frac{n}{3}}$, $z = 2^{6k+5}$ satisfy the given equation since $\left(2^{\frac{n}{2}}\right)^2 + \left(2^{\frac{n}{3}}\right)^3 = 2 \cdot 2^n = 2^{n+1} = \left(2^{6k+5}\right)^{2015}$. As k is arbitrary, one can find infinitely many solutions this way.

Remark. For finding Solution 1, one may search for solutions of the form $x = 2^{\frac{n}{2}}$, $y = 2^{\frac{n}{3}}$, $z = 2^{\frac{n+1}{2015}}$. For the solutions to be integral, n must be a multiple of 6 and $n+1$ must be a multiple of 2015. This leads the congruence system $n \equiv 0 \pmod{6}$, $n \equiv -1 \pmod{2015}$. As $\gcd(6, 2015) = 1$, the system has solutions by the Chinese remainder theorem. One of them is the value used in Solution 1.

Solution 2. Let a, b be positive integers such that $\gcd(a, b) = 1$. Define $c = a^2 + b^3$ and let $c = p_1^{\alpha_1} \cdot \ldots \cdot p_n^{\alpha_n}$ be its canonical representation. Let $k_i = 2015m_i - 336\alpha_i$, $i = 1, \ldots, n$, where m_i is an arbitrary positive integer satisfying $2015m_i - 336\alpha_i \geqslant 0$. Then the numbers $x = ap_1^{3k_1} \cdot \ldots \cdot p_n^{3k_n}$, $y = bp_1^{2k_1} \cdot \ldots \cdot p_n^{2k_n}$, $z = p_1^{6m_1-\alpha_1} \cdot \ldots \cdot p_n^{6m_n-\alpha_n}$ satisfy the given equation.

Remark. The solutions found in Solution 1 correspond to $a = b = 1$.

F15. *(Grade 12.)* Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the identity $f(f(xy)) = |x| f(y) + 3f(xy)$ for all real numbers x and y.

Answer: $f(x) = 0$, $f(x) = 4|x|$ and $f(x) = -4|x|$.

Solution 1. Interchanging x and y in given equation, we get $f(f(yx)) = |y| f(x) + 3f(yx)$. As $xy = yx$, together with the original equation it implies $|x| f(y) = |y| f(x)$. Substituting $x = 1$ to this gives $f(y) = |y| f(1)$, which must hold for every real number y. On the other hand, taking $x = y = 1$ in the original equation gives $f(f(1)) = f(1) + 3f(1) = 4f(1)$. Substituting $y = f(1)$ into the previous equation gives $f(f(1)) = |f(1)| f(1)$. Altogether, we get $4f(1) = |f(1)| f(1)$, i.e., $f(1)(|f(1)| - 4) = 0$. This leads to cases $f(1) = 0$, $f(1) = 4$ and $f(1) = -4$, which due to $f(y) = |y| f(1)$ give $f(x) = 0$, $f(x) = 4|x|$ and $f(x) = -4|x|$ as the possible solutions to the original equation. An easy check convinces that all three satisfy the equation.

Solution 2. For proving $f(y) = |y| f(1)$, we could also start with taking $x = 1$ in the original equation. This would establish $f(f(y)) = f(y) + 3f(y) = 4f(y)$ for all real numbers y. Using this result in the l.h.s. of the original equation, we get $4f(xy) = |x| f(y) + 3f(xy)$ and therefore $f(xy) = |x| f(y)$ for all x and y. Finally $y = 1$ gives $f(x) = |x| f(1)$ which is equivalent to the desired identity. The rest can be done as in Solution 1.

Solution 3. For obtaining the identity $f(y) = |y|f(1)$, we could also proceed as follows. Substituting $x = y = 0$ into the original equation gives $f(f(0)) = 3f(0)$. Substituting $x = 1, y = 0$ into the original equation gives $f(f(0)) = 4f(0)$. Hence $f(0) = 0$. Furthermore, taking $y \neq 0$, $x = \frac{1}{y}$ in the original equation gives $f(f(1)) = \frac{1}{|y|}f(y) + 3f(1)$, implying that $f(y) = |y| \cdot (f(f(1)) - 3f(1))$. Hence in the case $y \neq 0$ we have $f(y) = A|y|$ where A is some constant. This form extends to the case $y = 0$ because $f(0) = 0$. Now $f(y) = |y|f(1)$ results from taking $x = y = 1$ in the original equation, since $f(f(1)) = 4f(1)$ implies $A = f(f(1)) - 3f(1) = f(1)$.

Remark: The equality $f(0) = 0$ can be obtained also as follows. Taking $y = 0$ in the original equation gives $f(f(0)) = |x|f(0) + 3f(0)$, meaning that $|x|f(0) = f(f(0)) - 3f(0)$ for every real number x. In the case $f(0) \neq 0$, the l.h.s. would obtain all real values while the r.h.s. would be a constant. This contradiction shows that $f(0)$ must be zero.

F16. *(Grade 12.)* Let R and r be the circumradius and inradius, respectively, of a right triangle. Prove that $R \geqslant (1 + \sqrt{2})r$.

Solution 1. Let ABC be the given triangle with right angle at vertex C. Let its circumcentre and incentre be O and I, respectively. Then O coincides with the midpoint of hypotenuse AB. Let K, L, and M be the tangent points of the incircle with sides BC, CA, and AB, respectively. Let N be the point of intersection of ray CI with hypotenuse AB (Fig. 15). As both IK and AC are perpendicular with line BC, we have $IK \parallel LC$. As both IL and BC are perpendicular to line AC, we also have $IL \parallel KC$. In addition, $|IK| = |IL| = r$, whence $IKCL$ is a square with side length r and diagonal CI. Consequently, $\angle ACN = \angle BCN = 45^\circ$ and $|CI| = \sqrt{2}r$. Thus $|CI| + |IM| = \sqrt{2}r + r$. It remains to show that line segment CO is at least as long as broken line CIM.

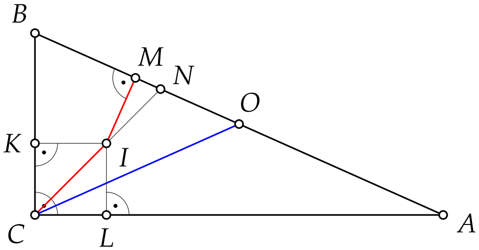

Fig. 15

Assume w.l.o.g. that $\angle BAC \leqslant \angle ABC$. Then $\angle BAC \leqslant 45^\circ$, implying $\angle AOC = 180^\circ - \angle ACO - \angle OAC = 180^\circ - \angle BAC - \angle BAC \geqslant 180^\circ - 45^\circ - \angle BAC = 180^\circ - \angle ACN - \angle NAC = \angle ANC$ and $\angle ANC = 180^\circ - \angle ACN - \angle NAC \geqslant 180^\circ - 45^\circ - 45^\circ = 90^\circ$. The inequalities $\angle AOC \geqslant \angle ANC \geqslant 90^\circ$ show that O lies on the line segment NA and triangle ANC has obtuse angle at vertex N (in the extreme case, right angle). Thus $|CO| \geqslant |CN| = |CI| + |IN| \geqslant |CI| + |IM|$, q.e.d.

Solution 2. Denote points O, I, K, L, M as in Solution 1. Also prove similarly to Solution 1 that $IKCL$ is a square with side length r. The area of triangle ABC equals the sum of the areas of right triangles IAL, IAM, IBM and

IBK and square $IKCL$. As $|AL| = |AM|$ and $|BM| = |BK|$, as well as $|IK| = |IL| = |IM| = r$, triangles IAL and IAM have equal areas and so have triangles IBM and IBK (Fig. 16). As the areas of triangles IAM and IBM sum up to the area of triangle IAB which is $\frac{1}{2}|AB|r$ and equals to Rr, we obtain $S = 2 \cdot Rr + r^2$. On the other hand, the area of triangle ABC equals the sum of isosceles triangles OAC and OBC, giving $S = \frac{1}{2}R^2 \sin\angle AOC + \frac{1}{2}R^2 \sin\angle BOC \leqslant \frac{1}{2}R^2 + \frac{1}{2}R^2 = R^2$. Therefore, $R^2 \geqslant 2Rr + r^2$. Hence $(R-r)^2 \geqslant 2r^2$, i.e., $R - r \geqslant \sqrt{2}r$. This provides us the desired inequality.

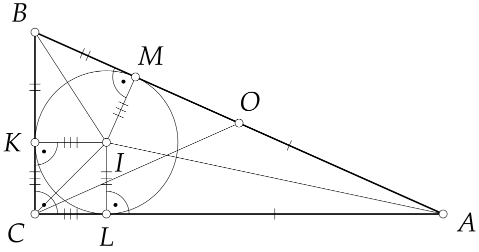

Fig. 16

F17. *(Grade 12.)* Let $x_0, \ldots, x_{n-1}$ be distinct real numbers such that $0 < |x_0| \leqslant \ldots \leqslant |x_{n-1}|$. Prove that the sums of elements of all subsets of $\{x_0, \ldots, x_{n-1}\}$ are 2^n consecutive members of an arithmetic sequence in some order if and only if the pairwise ratios of the numbers $|x_0|, \ldots, |x_{n-1}|$ are equal to those of the numbers $2^0, \ldots, 2^{n-1}$.

Solution. Denote $X = \{x_0, \ldots, x_{n-1}\}$. Assume that the sums of elements of its subsets form an arithmetic sequence and let $d > 0$ be the difference of the sequence. We prove by induction on n that $|x_i| = d \cdot 2^i$ for $0 \leqslant i < n$. The claim holds trivially for $n = 1$. Assume now that the claim holds for $n - 1$ numbers. Let s be the smallest among the 2^n sums and s' be the second smallest. The sum s is obviously obtained by the subset N of all negative elements of X. In order to obtain s' as the sum, some (possibly 0) negative elements are excluded and some (possibly 0) positive elements are included. It can be easily verified that the only possibility is either just excluding x_0 from N if $x_0 \in N$ or just including x_0 into N if $x_0 \notin N$ (other changes would increase the sum more). Hence $d = s' - s = |x_0|$. Now leave x_0 out from X. Exactly half of all sums remain; other sums differ from corresponding sums by $|x_0|$ to the same direction. As $d = |x_0|$, this means that, in the arithmetic sequence of sums, exactly one member out of every two consecutive members is dropped. Thus the result is an arithmetic sequence with difference $2d$. By the induction hypothesis, $|x_i| = 2d \cdot 2^{i-1} = d \cdot 2^i$ for $1 \leqslant i < n$. Including $|x_0| = d = d \cdot 2^0$ completes the proof.

For proof in the other direction, let $|x_i| = d \cdot 2^i$ for $0 \leqslant i < n$. Consider two different subsets of X; suppose their elements sum up to the same number. Assume w.l.o.g. that these two subsets are disjoint. Then every x_k that occurs in one or another subset can be expressed as a linear combination of others with coefficients 1 and -1. But if x_k is the largest by absolute value term occurring in these two subsets then this is impossible, since $|x_k| = d \cdot 2^k > d \cdot (2^0 + \ldots + 2^{k-1}) = |x_0| + \ldots + |x_{k-1}|$. Consequently, all subsets of X have different sums of elements. Let s and t be the sums of all negative and all positive elements of X, respectively. Clearly, s is the smallest and t is the largest sum of elements of a subset. Their difference is $t - s = |x_0| + \ldots + |x_{n-1}| = d \cdot (2^0 + \ldots + 2^{n-1}) = d \cdot (2^n - 1)$. As all sums of elements of subsets are integral multiples of d, this implies that exactly 2^n of them lie on the interval $[s, t]$. By pairwise distinctness, sums of elements of all subsets of X cover all integral multiples of d between s and t, hence forming an arithmetic sequence.

F18. *(Grade 12.)* Prove that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2015} > \frac{13}{2}$.

Solution 1. Denote $H(k) = \frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}}$ and use the estimation $H(k) > 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}$ for all $k \geqslant 2$. This gives $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2015} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + H(2) + H(3) + \ldots + H(10) - \left(\frac{1}{2016} + \frac{1}{2017} + \ldots + \frac{1}{2048}\right) > \left(4 \cdot \frac{1}{2} + \frac{1}{12}\right) + \left(9 \cdot \frac{1}{2}\right) - \left(33 \cdot \frac{1}{2016}\right) = \frac{13}{2} + \frac{1}{12} - \frac{33}{2016} > \frac{13}{2}$.

Solution 2. As the function $y = \frac{1}{x}$ is decreasing we have $\frac{1}{n} > \int_n^{n+1} \frac{1}{x} \mathrm{d}x$ for every positive integer n. Hence $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2015} > \int_1^2 \frac{1}{x} \mathrm{d}x + \int_2^3 \frac{1}{x} \mathrm{d}x + \int_3^4 \frac{1}{x} \mathrm{d}x + \ldots + \int_{2015}^{2016} \frac{1}{x} \mathrm{d}x = \int_1^{2016} \frac{1}{x} \mathrm{d}x$. By calculus, $\int_1^{2016} \frac{1}{x} \mathrm{d}x = \ln 2016 - \ln 1 = \ln 2016$. However, $\ln 2016 > \frac{13}{2}$ since $\ln 2016 > \log_3 2016 > \log_3 1458 = 6 + \log_3 2 > 6 + \frac{1}{2}$, where the last inequality holds because $2 > \sqrt{3}$.

IMO Team Selection Contest I

S1. Let n be a natural number, $n \geqslant 5$, and $a_1, a_2, \ldots, a_n$ real numbers such that all possible sums $a_i + a_j$, where $1 \leqslant i < j \leqslant n$, form $\frac{n(n-1)}{2}$ consecutive members of an arithmetic progression when taken in some order. Prove that $a_1 = a_2 = \ldots = a_n$.

Solution. Let d be the difference of successive members of this arithmetic progression of pairwise sums. We show that the conditions force d to be 0; this also lets us conclude that $a_1 = a_2 = \ldots = a_n$, because for example $a_1 + a_3 = a_2 + a_3$, which gives $a_1 = a_2$, and for every $i = 2, 3, \ldots, n-1$, $a_1 + a_i = a_1 + a_{i+1}$, which gives $a_i = a_{i+1}$.

Without loss of generality let $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ and $d \geqslant 0$. The least sum of the ones listed is $a_1 + a_2$. The next one is $a_1 + a_3$, because for every $i > 3$, $a_1 + a_i \geqslant a_1 + a_3$ and for every $i > 1$ and $j > i$, $a_i + a_j \geqslant a_1 + a_3$. Therefore $(a_1 + a_3) - (a_1 + a_2) = d$, giving $a_3 - a_2 = d$. Analogously, the greatest sums are $a_n + a_{n-1}$ and $a_n + a_{n-2}$, giving $a_{n-1} - a_{n-2} = d$. If $n \geqslant 6$, then $2, 3, n-2, n-1$ are pairwise different indices, which means that $a_2 + a_{n-1}$ and $a_3 + a_{n-2}$ are different elements of this arithmetic progression. But since $a_3 + a_{n-2} = (a_2 + d) + (a_{n-1} - d) = a_2 + a_{n-1}$, this means that the arithmetic progression is constant, i.e., $d = 0$. If $n = 5$, then according to results above, $a_3 = a_2 + d$ and $a_4 = a_3 + d$. Assume $d > 0$. Since $(a_1 + a_3) - (a_1 + a_2) = (a_1 + a_4) - (a_1 + a_3) = d$, the sums $a_1 + a_2$, $a_1 + a_3$, and $a_1 + a_4$ are the three smallest ones among all pairwise sums. Analogously, $a_2 + a_5$, $a_3 + a_5$, and $a_4 + a_5$ are the three greatest sums. We notice that $a_2 + a_3$, $a_2 + a_4$, and $a_3 + a_4$ are three consecutive sums, because $a_2 + a_4 = a_3 + a_3$. Therefore, there are only two possibilities of how the sums can be ordered: $a_1 + a_2 < a_1 + a_3 < a_1 + a_4 < a_1 + a_5 < a_2 + a_3 < a_2 + a_4 < a_3 + a_4 < a_2 + a_5 < a_3 + a_5 < a_4 + a_5$ and $a_1 + a_2 < a_1 + a_3 < a_1 + a_4 < a_2 + a_3 < a_2 + a_4 < a_3 + a_4 < a_1 + a_5 < a_2 + a_5 < a_3 + a_5 < a_4 + a_5$. In the first case we conclude from the forms of the four least sums that $a_5 - a_4 = d$, which gives $a_2 + a_5 = a_3 + a_4$, a contradiction. In the second case, we analogously get from the forms of four greatest sums that $a_2 - a_1 = d$, whence $a_1 + a_4 = a_2 + a_3$, again a contradiction. Therefore $d = 0$, as claimed.

S2. A square-shaped pizza with side length 30 cm is cut into pieces (not necessarily rectangular). All cuts are parallel to the sides, and the total length of the cuts is 240 cm. Show that there is a piece whose area is at least 36 cm^2.

Solution 1. Let $S_1, \ldots, S_n$ be the areas of the pieces, and $p_1, \ldots, p_n$ their perimeters. Let the mth piece have the largest area, i.e., $S_i \leqslant S_m$ for all $i = 1, \ldots, n$. It suffices to show that $S_m \geqslant 36$ cm^2. As each cut gives rise to edges of pieces in both sides and the edges of the pizza are also contributing to the total length of the edges, we have $p_1 + \ldots + p_n = 2 \cdot 240 \text{ cm} + 4 \cdot 30 \text{ cm} = 600$ cm. For every $i = 1, \ldots, n$, let a_i and b_i be the side lengths of the smallest rectangle with sides parallel to the edges of the pizza and containing the ith piece. By AM-GM, $\sqrt{S_i} \leqslant \sqrt{a_i b_i} \leqslant \frac{a_i + b_i}{2} \leqslant \frac{p_i}{4}$. Hence

$$900 \text{ cm}^2 = S_1 + \ldots + S_n = \sqrt{S_1}\sqrt{S_1} + \ldots + \sqrt{S_n}\sqrt{S_n}$$
$$= \sqrt{S_m} \cdot \left(\sqrt{S_1} + \ldots + \sqrt{S_n}\right) \leqslant \sqrt{S_m} \cdot \left(\frac{p_1}{4} + \ldots + \frac{p_n}{4}\right) = \sqrt{S_m} \cdot 150 \text{ cm}.$$

This directly implies $\sqrt{S_m} \geqslant 6$ cm, i.e., $S_m \geqslant 36$ cm^2, q.e.d.

Solution 2. As in Solution 1, denote the perimeters and areas by $p_1, \ldots, p_n$ and $S_1, \ldots, S_n$ and prove the equality $p_1 + \ldots + p_n = 600$ cm. By conditions, the total area of the pieces equals 900 cm^2. Let $\frac{S_m}{p_m}$ be the largest among $\frac{S_1}{p_1}, \ldots, \frac{S_n}{p_n}$. Then $\frac{S_m}{p_m} \cdot p_i \geqslant S_i$ for every $i = 1, \ldots, n$; adding all these inequalities gives $\frac{S_m}{p_m} \cdot (p_1 + \ldots + p_n) \geqslant S_1 + \ldots + S_n$, implying $\frac{S_m}{p_m} \geqslant \frac{S_1 + \ldots + S_n}{p_1 + \ldots + p_n} = \frac{3}{2}$ cm. On the other hand, we show that a piece with fixed perimeter has the largest area if it is a square. If the piece has not rectangular shape then the rectangle whose both horizontal and vertical coverage equals that of the piece under consideration obviously has the same or smaller perimeter but larger area. In the case of smaller perimeter, the rectangle can be transformed to a similar rectangle so that the area becomes even larger. Hence we may assume w.l.o.g. that the pieces have rectangular shape. If the side lengths are a and b then the side length of a square with the same perimeter is $\frac{a+b}{2}$. By AM-GM, we have $\left(\frac{a+b}{2}\right)^2 \geqslant ab$, meaning that the area of such a square is at least as large as the area of the rectangle. A square with perimeter p_i has side length $\frac{p_i}{4}$. Thus $\frac{S_i}{p_i} \leqslant \frac{\left(\frac{p_i}{4}\right)^2}{p_i} = \frac{p_i}{16}$, whence $S_i \leqslant \frac{p_i^2}{16}$, i.e., $p_i \geqslant 4\sqrt{S_i}$. Supposing that all pieces have area less than 36 cm^2 would give $\frac{3}{2}$ cm $\leqslant \frac{S_m}{p_m} \leqslant \frac{S_m}{4\sqrt{S_m}} = \frac{\sqrt{S_m}}{4} < \frac{6\text{ cm}}{4} = \frac{3}{2}$ cm, a contradiction.

Remark. The auxiliary result $\frac{S_m}{p_m} \geqslant \frac{S_1+\ldots+S_n}{p_1+\ldots+p_n}$ is actually the well-known *mediant inequality* for fractions $\frac{S_1}{p_1}, \ldots, \frac{S_m}{p_m}$. It can also be established by considering the weighted mean of these fractions with weights $\frac{p_1}{p_1+\ldots+p_n}, \ldots, \frac{p_n}{p_1+\ldots+p_n}$.

Solution 3. As in Solution 1, denote the perimeters and areas by $p_1, \ldots, p_n$ and $S_1, \ldots, S_n$ and prove the equality $p_1 + \ldots + p_n = 600$ cm. Suppose that all pieces have area less than 36 cm^2. Then $n > 25$ because otherwise the total area would be less than 900 cm^2, contradicting the assumptions. We shall transform the collection of the pieces by the following phases; note that the sum of the perimeters of the pieces does not increase.

1) Replace all non-rectangular pieces with rectangles with the same area. For that, replace each piece with a rectangle with both horizontal and vertical coverage being the same, and then transform the result to a similar rectangle to the initial area. Neither step can increase the perimeter.

2) Replace all rectangles that are not squares with squares with the same area. Let the area of a rectangular piece be S and side lengths x and $\frac{S}{x}$. The perimeter is $p(x) = 2 \cdot \left(x + \frac{S}{x}\right)$, whence $p'(x) = 2 \cdot \left(1 - \frac{S}{x^2}\right)$. The derivative is zero iff $S = x^2$. The extremum found can only be minimum.

3) While the collection of pieces contains squares with area less than 36 cm^2, choose two of them and increase the larger and decrease the smaller by equal amount so that the area of the large would be as close as possible to 36 cm^2; if the area of the smaller one would become zero then leave this piece out of the collection. Let the total area of two squares chosen be $2S$ while the area of the smaller square be x; then the sum of their perimeters is $q(x) = 4\sqrt{x} + 4\sqrt{2S - x}$. The derivative $q'(x) = \frac{2}{\sqrt{x}} - \frac{2}{\sqrt{2S-x}}$ is zero in the case $x = S$ and is positive in the interval $0 < x < S$. Thus the total perimeter is the smaller, the smaller is the smaller square.

As the area of any piece can exceed 36 cm^2 and the total area is 900 cm^2, this process ends in a state with exactly 25 square-shaped pieces, each of area 36 cm^2. Their sum of perimeters is 600 cm. As this quantity could not increase during the process, the initial sum of perimeters had to be larger, contradicting the equality $p_1 + \ldots + p_n = 600$ cm.

S3. Let q be a fixed positive rational number. Call number x *charismatic* if there exist a positive integer n and integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$x = (q+1)^{\alpha_1} \cdot (q+2)^{\alpha_2} \cdot \ldots \cdot (q+n)^{\alpha_n}.$$

a) Prove that q can be chosen in such a way that every positive rational number turns out to be charismatic.

b) Is it true for every q that, for every charismatic number x, the number $x+1$ is charismatic, too?

Answer: b) no.

Solution. a) Take $q = 1$ and let x be any positive rational number. Let $n = p - 1$, where p is the largest prime number that divides either the numerator or the denominator of x. Then all prime numbers occurring in the canonical representation of x with non-zero exponent are in the form $1+i$ with $1 \leqslant i \leqslant n$. In order to obtain a product required in the definition of charismaticity, equip such prime numbers with their exponent in the canonical representation of x and take all other exponents α_i to be zero.

b) Take $q = \frac{1}{3}$ and $x = 1$. Since $1 = (q+1)^0$, the number x is charismatic. Suppose that 2 is charismatic. Then there exist a positive integer n and integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\left(\frac{1}{3}+1\right)^{\alpha_1}\left(\frac{1}{3}+2\right)^{\alpha_2} \ldots \left(\frac{1}{3}+n\right)^{\alpha_n} = 2$. This is equivalent to $(3 \cdot 1+1)^{\alpha_1}(3 \cdot 2+1)^{\alpha_2} \ldots (3 \cdot n+1)^{\alpha_n} = 2 \cdot 3^{\alpha_1+\alpha_2+\ldots+\alpha_n}$. Obviously $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 0$, since the bases of powers in the l.h.s. are not divisible by 3 and therefore 3 does not occur in the canonical representation of the product of these powers. Thus $(3 \cdot 1+1)^{\alpha_1}(3 \cdot 2+1)^{\alpha_2} \ldots (3 \cdot n+1)^{\alpha_n} = 2$. Let the positive exponents be $\alpha_{i_1}, \ldots, \alpha_{i_k}$ and the negative exponents be $\alpha_{j_1}, \ldots, \alpha_{j_l}$. The condition obtained is equivalent to

$$\frac{(3i_1+1)^{\alpha_{i_1}} \ldots (3i_k+1)^{\alpha_{i_k}}}{(3j_1+1)^{|\alpha_{j_1}|} \ldots (3j_l+1)^{|\alpha_{j_l}|}} = 2,$$

which is in turn equivalent to $(3i_1+1)^{\alpha_{i_1}} \ldots (3i_k+1)^{\alpha_{i_k}} = 2 \cdot (3j_1+1)^{|\alpha_{j_1}|} \ldots (3j_l+1)^{|\alpha_{j_l}|}$. The l.h.s. and r.h.s. of this equality are congruent to 1 and 2 modulo 3, respectively. The contradiction shows that 2 is not charismatic, whence the condition checked is not true.

S4. Altitudes AD and BE of an acute triangle ABC intersect at H. Let P be the point of tangency of the circle with radius HE centered at H with its tangent line going through point C, and Q be the point of tangency of the circle with radius BE centered at B with its tangent line going through point C. Prove that points D, P, and Q are collinear.

Solution 1. If $|AC| = |BC|$ then $P = D$ and the claim to be proven is trivial (Fig. 17). In the following we assume that $|AC| \neq |BC|$. By construction the points E and Q are the reflections of each other from the line CB. Since D is located on the line CB, we conclude that $\angle CDQ = \angle CDE$. On the other hand, conditions of the problem give $\angle CDH = \angle CEH = \angle CPH = 90^\circ$,

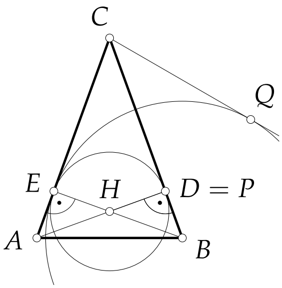

Fig. 17

therefore points C, H, D, E, and P are located on a circle with diameter CH. Because of the property of tangent lines, $|CP| = |CE|$, and we conclude that CP and CE are chords of equal lengths of the circle with diameter CH. If $|AC| > |BC|$ (Fig. 18), then the point D is located towards the midpoint of the circle from both chords, therefore $\angle CDE = \angle CDP$. Together with the equality $\angle CDQ = \angle CDE$ this gives that points D, P and Q are collinear. If $|AC| < |BC|$ (Fig. 19), then the point D is located on the same side as the center of the circle with respect to the chord CE, but on the other side with respect to the chord CP, which means $\angle CDE = 180° - \angle CDP$. Together with $\angle CDQ = \angle CDE$ this again gives that points D, P and Q are collinear.

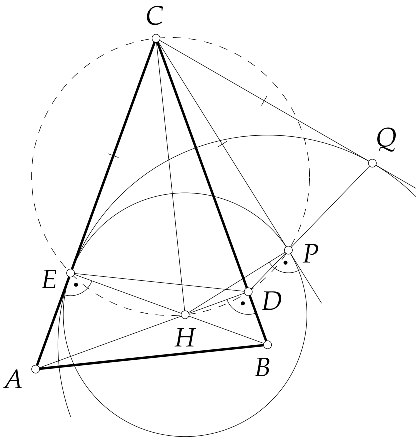

Fig. 18

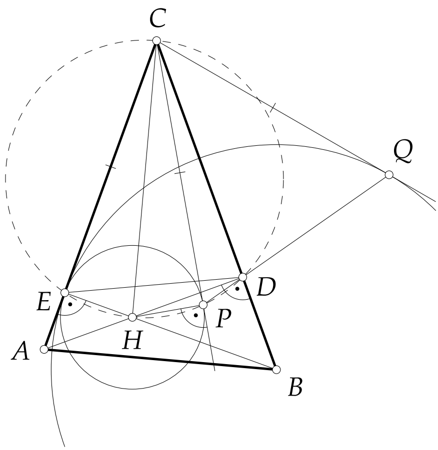

Fig. 19

Solution 2. Assume that $|AC| \neq |BC|$. According to the construction, points P and E are the reflections of each other from the line CH, and Q and E are the reflections of each other from the line CB. Therefore $|CP| = |CE| = |CQ|$ and $\angle ECH = \angle PCH$ and $\angle ECB = \angle QCB$. By using the fact that triangle PCQ is isosceles and the equation $\angle PCQ = \angle ECQ - \angle ECP$, we get that $\angle CPQ = 90° - \frac{\angle PCQ}{2} = 90° - \left(\frac{\angle ECQ}{2} - \frac{\angle ECP}{2}\right) = 90° - (\angle ECD - \angle ECH) = 90° - \angle HCD$. Since $\angle HDC = \angle HPC = 90°$, the points C, H, D, P are located on a circle with diameter CH. If now $|AC| > |BC|$, then from the inscribed angles we get that $\angle HPD = \angle HCD$ and $\angle HPD + \angle HPC + \angle CPQ = \angle HCD + 90° + (90° - \angle HCD) = 180°$. So the points D, P and Q are collinear. If $|AC| < |BC|$, then we get that $\angle HPD = 180° - \angle HCD$ and $\angle CPD = \angle HPD - \angle HPC = (180° - \angle HCD) - 90° = \angle CPQ$. From here, again, D, P and Q are collinear.

Remark. This problem originally appeared in the final round of Ukrainian Mathematical Olympiad 2015.

S5. Find all functions f from reals to reals which satisfy $f(f(x)+f(y)) = f(x^2)+2x^2f(y)+(f(y))^2$ for all real numbers x and y.

Answer: $f(x)=0$ and $f(x)=x^2$.

Solution. By switching the variables x and y in the original equation, we obtain $f\big(f(y)+f(x)\big) = f(y^2)+2y^2f(x)+f(x)^2$. Since $f(x)+f(y) = f(y)+f(x)$, the l.h.s. of this equation is equal to the l.h.s. of the original equation. Then also the right hand sides must be equal, i.e., $f(x^2)+2x^2f(y)+f(y)^2 = f(y^2)+2y^2f(x)+f(x)^2$. By taking $y=0$ here and denoting $f(0)=a$, we obtain $f(x^2)+2x^2a+a^2 = a+f(x)^2$. By instead taking $y=1$ and denoting $f(1)=b$, we get $f(x^2)+2x^2b+b^2 = b+2f(x)+f(x)^2$. By subtracting the previous equation form the last equation, we obtain $2(b-a)x^2+b^2-a^2 = b-a+2f(x)$.

By taking $x=0$ here, we get $b^2-a^2=b+a$. Since $b^2-a^2=(b+a)\cdot(b-a)$, there are two possibilities: either $b+a=0$ or $b-a=1$. 1) If $b+a=0$, then substituting $x=0$ and $y=1$ into the original equation gives $f(0)=f(a+b)=a+b^2$, from which we conclude that $b^2=0$, meaning $b=0$ and also $a=0$. From the last equation for $f(x)$ we now get that $f(x)=0$ for every real number x. 2) If $b-a=1$ or in other words $b=a+1$, then the equation for $f(x)$ gives $2f(x)=2x^2+(b+a-1)=2x^2+2a$ for every real number x, so $f(x)=x^2+a$, from where in case $x=2a$ we get $f(2a)=4a^2+a$. By taking $x=y=0$ in the original equation, we get $f(2a)=a+a^2$. Therefore $a^2=4a^2$, whence $a=0$. So $f(x)=x^2$ for every real number x. A check confirms that the functions $f(x)=0$ and $f(x)=x^2$ do satisfy the original equation.

S6. In any rectangular game board with black and white squares, call a row X a *mix* of rows Y and Z whenever each cell in row X has the same colour as either the cell of the same column in row Y or the cell of the same column in row Z. Let a natural number $m \geq 3$ be given. In some rectangular board, black and white squares lie in such a way that all the following conditions hold.

1) Among every three rows of the board, one is a mix of two others.
2) For every two rows of the board, their corresponding cells in at least one column have different colours.
3) For every two rows of the board, their corresponding cells in at least one column have equal colours.
4) It is impossible to add a new row with each cell either black or white to the board in a way leaving both conditions 1) and 2) still in force.

Find all possibilities of what can be the number of rows of the board.

Answer: 4.

Solution. At first we show that a board with 4 rows is always possible. Colour the first two cells in all possible fashions (WW, WB, BW, BB) and let all other cells be white. Condition 2) holds by construction. Condition 3) holds as $m \geq 3$ whence the last cell has the same colour in all rows. Condition 1) holds, because among every three rows, some two have cells of opposite colours in both of the first two columns, whence the remaining row is a mix of them. For condition 4), suppose a row U is added to the board. If the last $m - 2$ cells of U are white, condition 2) is violated. Otherwise, U cannot be a mix of any two rows the board initially had. Assuming still condition 1), some row in each pair of old rows must be a mix of U and the other row in the pair. If the two old rows have cells of opposite colours in both of the first two columns, the corresponding cells in U must be coloured similarly to the row which is a mix of U and the other one. This leads to contradiction, since there are two disjoint pairs of old rows with cells of opposite colours in both of the first two columns.

In the rest, we show that the number of rows can be nothing else but 4. Let k be the largest number of cells by which some two rows of the given board differ; let A and B be rows with exactly k different cells. By condition c), $k < m$. As inverting all colours in one column preserves conditions 1)–4), assume w.l.o.g. that all cells of row A are white and row B contains exactly k black cells. Likewise, conditions 1)–4) are preserved under interchanging of columns, whence assume w.l.o.g. that the first k cells of B are black and the remaining $m - k$ cells are white.

Let X be a row having black cells in the last $m - k$ columns. If the first k cells of X are black then A and X differ from each other by more than k squares. If the first k cells of X are white then B and X differ from each other by more than k squares. Both situations are impossible by choice of A and B. If some of the first k cells in X are white and some are black, none of rows A, B, X is a mix of the other two, which contradicts condition 1). Consequently, there is no row like X, i.e., the last $m - k$ columns consist entirely of white squares.

For each row R, denote by BC R the set of all columns whose cell in row R is black. Suppose that for every two rows X and Y, either $\mathrm{BC}\,X \subseteq \mathrm{BC}\,Y$ or $\mathrm{BC}\,Y \subseteq \mathrm{BC}\,X$. After adding an entirely black new row T to the board, condition 2) still holds. Taking any two old rows X and Y and assuming w.l.o.g. that $\mathrm{BC}\,X \subseteq \mathrm{BC}\,Y$, row Y is a mix of rows X and T. Hence the board with the new row satisfies also condition 1). This contradicts condition 4). Thus there exist rows X and Y such that $\mathrm{BC}\,X \nsubseteq \mathrm{BC}\,Y$ and $\mathrm{BC}\,Y \nsubseteq \mathrm{BC}\,X$.

This means that, in some column, the cell of X is black and the cell of Y is white, while in another column, the cell of Y is black and the cell of X is white. Thus X is neither a mix of A and Y nor a mix of Y and B, as well as Y is neither a mix of A and X nor a mix of X and B. By condition 1), both A and B must be mixes of X and Y. This means that in each of the first k columns, the cells of X and Y have opposite colours.

Suppose that the board contains some row Z in addition to A, B, X, Y. By condition 2), Z is coloured differently from these four rows. Hence either $\mathrm{BC}\,Z \subseteq \mathrm{BC}\,X$ or $\mathrm{BC}\,X \subseteq \mathrm{BC}\,Z$, because otherwise Z should be coloured the same way as Y according to the previous paragraph. Similarly also $\mathrm{BC}\,Z \subseteq \mathrm{BC}\,Y$ or $\mathrm{BC}\,Y \subseteq \mathrm{BC}\,Z$. But $\mathrm{BC}\,Z \subseteq \mathrm{BC}\,X$ and $\mathrm{BC}\,Y \subseteq \mathrm{BC}\,Z$ together imply $\mathrm{BC}\,Y \subseteq \mathrm{BC}\,X$ which is impossible as $\mathrm{BC}\,X$ and $\mathrm{BC}\,Y$ are disjoint and non-empty. Analogously, combining $\mathrm{BC}\,Z \subseteq \mathrm{BC}\,Y$ and $\mathrm{BC}\,X \subseteq \mathrm{BC}\,Z$ leads to contradiction. Thus either $\mathrm{BC}\,Z \subseteq \mathrm{BC}\,X$ and $\mathrm{BC}\,Z \subseteq \mathrm{BC}\,Y$ or $\mathrm{BC}\,X \subseteq \mathrm{BC}\,Z$ and $\mathrm{BC}\,Y \subseteq \mathrm{BC}\,Z$; in the first case, Z is coloured like A, and in the second case, Z is coloured like B. Hence such a row Z does not exist and the number of rows equals 4.

IMO Team Selection Contest II

S7. Prove that for every prime number p and positive integer a, there exists a natural number n such that p^n contains a consecutive equal digits.

Solution. For every integer x and a positive integer m that is relatively prime with x, there exists a positive integer t such that $x^t \equiv 1 \pmod{m}$. If $p \neq 2$ and $p \neq 5$, then for every natural number k, there exists some integer $t > 0$ satisfying $p^t \equiv 1 \pmod{10^k}$. As obviously $p^t > 10^k$, the number p^t ends with $k-1$ zeros followed by one. Thus choosing $k > a$ and $n = t$ solves the problem. If $p = 2$ or $p = 5$, then denote $q = \frac{10}{p}$. For every natural number k, there exists some integer $t > 0$ satisfying $p^t \equiv 1 \pmod{q^k}$. This implies $p^{k+t} \equiv p^k \pmod{10^k}$. As obviously $p^{k+t} > 10^k > p^k$, the ending digits of the number p^{k+t} are precisely all the digits of number p^k preceded by zeros until the kth digit from the end. Choose k in such a way that $q^k > 10^a$; then $p^k < 10^{k-a}$. This means that the number of consecutive zeros must be at least a and taking $n = k + t$ solves the problem.

S8. Find all positive integers n for which it is possible to partition a regular n-gon into triangles with diagonals not intersecting inside the n-gon such that at every vertex of the n-gon an odd number of triangles meet.

Answer: $3k$, where k is a positive integer.

Solution 1. Paint the triangles black and white with triangles on different sides of every diagonal having different colours. This is possible, because colouring a single triangle is always possible and by cutting a polygon into two with any diagonal, we can color each side independently and, if triangles that on two sides of that diagonal happen to have the same colour, swap the colours of all triangles on one side of the diagonal. When colouring like this the condition that at every vertex an odd number of triangles meet is equivalent to the condition that the triangles that share at least one side with the original polygon all have the same colour.

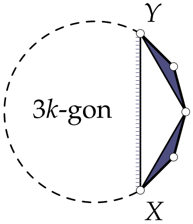

Fig. 20

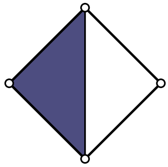

Fig. 21

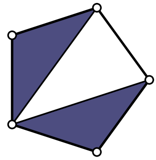

Fig. 22

We show by induction that if n is divisible by 3, then we can partition the polygon as needed. The statement is trivially true for $n = 3$. Every appropriately partitioned n-gon, where the triangles lining its sides are w.l.o.g. coloured black, can be turned to an appropriately partitioned $(n+3)$-gon by adding to any side XY a white triangle outside the polygon and to the two new sides in turn black triangles (Fig. 20).

Now consider any n-gon that has been partitioned in the required way. Assume that the polygons with less than n vertices can only be partitioned when then number of vertices is divisible by 3. First assume that the only diagonals that have been used for partitioning are such that there is only one vertex of the polygon on one or the other side of it. Such vertex cannot be the endpoint of any of the diagonals used in partitioning the polygon. Therefore there must be $n-3$ isolated points in the partition, because the number of diagonals is $n-3$. This means that all diagonals have been drawn between 3 different vertices. Because one can only draw a maximum of 3 different diagonals between 3 vertices, $n - 3 \leqslant 3$, which means $n \leqslant 6$. But in cases $n = 4$ and $n = 5$ w.l.o.g. there is only one way to partition the polygon with non-intersecting diagonals, which does not satisfy the conditions (Fig. 21 and 22).

It remains to analyse the case where at least one of the diagonals d used for partition has more than one vertex of the polygon on both sides of it. W.l.o.g. the polygon has only black triangles lining its sides. Cut the n-gon into two pieces via the diagonal d, the numbers of vertices of which are x and y. One of the polygons obtained must also have only black triangles lining its sides, let that be the one with x vertices, while the other one has black triangles along all of its sides except for the former diagonal d, where there is a white triangle. By adding another triangle to the outside of the y-gon adjacent to the side d, we add 1 vertex and again get a polygon the sides of which are the sides of just black triangles. From the induction hypothesis we get that x and $y+1$ are divisible by 3. Since $n = x + y - 2 = x + (y + 1) - 3$, n must also be divisible by 3.

Solution 2. First we construct a partition satisfying the conditions for every $n = 3k$. Label the vertices of the polygon $0, 1, \ldots, n-1$ in order of adjacency. Draw diagonals from vertex 0 to every one of its non-neighbouring vertices the number of which is not divisible by 3, and also connect vertices

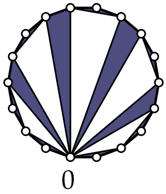

Fig. 23

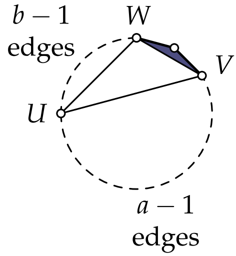

Fig. 24

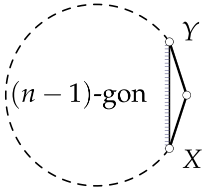

Fig. 25

$3i-1$ and $3i+1$ with a diagonal for every $i = 1, \ldots, k-1$ (Fig. 23 shows an example for $k = 6$). With this the n-gon has been partitioned into triangles, because the chosen diagonals do not intersect and their number is $2(k-1)+(k-1) = n-3$. To every vertex the number of which is divisible by 3 except for 0 and also to vertices 1 and $n-1$, 0 diagonals have been drawn; to every vertex the number of which is not divisible by 3, except for 1 and $n-1$, 2 diagonals have been drawn and to the vertex 0, $2(k-1)$ diagonals have been drawn. Since all these are even numbers, there is an odd number of triangles meeting at every vertex.

To show that a suitable partition only exists for the numbers divisible by 3, we interpret the condition by colouring the triangles and by the monochromatism of the contour of the polygon as in Solution 1. Consider an appropriate partition of an n-gon, where $n \geqslant 4$, and assume that the statement holds for the polygons with smaller number of vertices. W.l.o.g. the triangles sharing sides with the original polygon are black. Let U be any vertex where more than one triangle meet. Consider a white triangle one vertex of which is U; let the other vertices be V and W. Because the sides of the n-gon are only lined by black triangles, V and W are joined by a diagonal and not by a side of the n-gon. In the n-gon, the sides of triangle UVW each cut off a polygon that satisfies the conditions of the problem, because all its sides are lined with black triangles. By denoting the numbers of vertices of those polygons by x, y, and z, we have $x+y+z = n+3$, because any vertex of the polygon belongs to at least one of the three polygons and the vertices U, V, and W each belong to two polygons. Also, x, y, and z are all less than n, because for every diagonal forming the triangle UVW, at least the third vertex of the triangle UVW is located on the other side than the polygon being cut off by this diagonal. By the induction hypothesis, x, y and z are all divisible by 3 and therefore n also is.

Solution 3. Colour the triangles and interpret the condition of the problem as monochromatism of the contour of the n-gon as in previous solutions. We first show by induction on n that if $3 \mid n$, then the necessary partition exists. If $n = 3$, the statement is trivially true. If $n > 3$, then $n = a+b$, where $3 \mid a$, $3 \mid b$ and $3 \leqslant a < n$, $3 \leqslant b < n$. Pick a vertex U

of the n-gon and draw diagonals from V to vertices V, located $a-1$ sides counterclockwise from U, and to W, located $b-1$ sides clockwise from U. Vertices V and W are separated from each other by 2 sides; connect them with a diagonal as well (Fig. 24). Now the n-gon has been partitioned into two triangles, an a-gon, and a b-gon. Let the triangle UVW be white and the other triangle black; by induction hypothesis we can partition the a-gon and the b-gon into triangles with non-intersecting diagonals such that all of their sides are black. Together we obtain the necessary partition.

In the following we show that the condition $3 \mid n$ is also necessary for finding an appropriate partition. First we prove that in every such partitioning there exists a vertex at which only one partitioning triangle "meets". Indeed, as the n-gon only has n sides and every one of them is a side of some triangle used in partition, but there are only $n-2$ triangles in a partition, there must exist a triangle that is bordered by two sides of the polygon. The common vertex of those sides is the one we're looking for.

From the proven lemma we conclude that every partition into triangles with non-intersecting diagonals can be obtained by adding vertices one by one between two existing neighbouring vertices X and Y together with a new triangle (Fig. 25). As the colour of the new triangle is the opposite to the one of the triangle on the other side of the edge XY that previously existed, with every step the polygon obtains two new black edges instead of one white or two whites instead of one black. In both cases the difference between the number of white and black edges changes by 3. Because in the original triangle the numbers of edges coloured in one and the other way are 3 and 0, respectively, because of which the numbers of white and black edges are congruent modulo 3 at the beginning of the process, this property is also valid after every step of the process. In the partition satisfying conditions all n sides have the same colour, meaning $n \equiv 0 \pmod 3$. The desired result follows.

Remark. This problem was taken from the set of problems proposed to Baltic Way 2014 (and slightly modified).

S9. The orthocenter of an acute triangle ABC is H. Let K and P be the midpoints of lines BC and AH, respectively. The angle bisector drawn from the vertex A of the triangle ABC intersects with line KP at D. Prove that $HD \perp AD$.

Solution. Let B' and C' be the feet of the altitudes drawn from vertices B and C (Fig. 26). Because $\angle AB'H = \angle AC'H = 90°$, points B' and C' are located on a circle with diameter AH centered at P. Analogously, since $\angle BC'C = \angle CB'B = 90°$, points B' and C' are located on a circle with diameter BC centered at K. Because PK connects the midpoints of the circles and C' and B' are the points of intersection of both circles, line PK divides the arcs $B'C'$ of both circles into equal halves. The arc $B'C'$ on the circle with diameter AH is also divided in half by the angle bisector drawn from the vertex A. Therefore lines PK and the angle bisector drawn from A intersect on the circle with diameter AH. Consequently $\angle HDA = 90^\circ$.

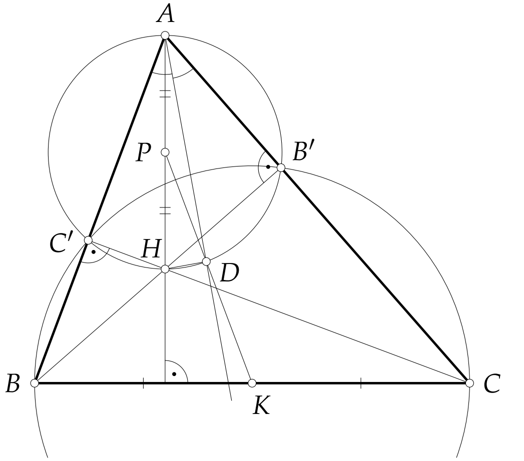

Fig. 26

S10. Let n be an integer and a, b real numbers such that $n > 1$ and $a > b > 0$. Prove that $(a^n - b^n)\left(\frac{1}{b^{n-1}} - \frac{1}{a^{n-1}}\right) > 4n(n-1)\left(\sqrt{a} - \sqrt{b}\right)^2$.

Solution. Denote $x = \sqrt{a}$, $y = \sqrt{b}$; then the given inequality reduces to

$$(x^{2n} - y^{2n})\left(\frac{1}{y^{2n-2}} - \frac{1}{x^{2n-2}}\right) > 4n(n-1)(x-y)^2.$$

Use factorizations $x^{2n} - y^{2n} = (x-y)\left(x^{2n-1} + x^{2n-2}y + \ldots + y^{2n-1}\right)$ and $\frac{1}{y^{2n-2}} - \frac{1}{x^{2n-2}} = \left(\frac{1}{y} - \frac{1}{x}\right)\left(\frac{1}{y^{2n-3}} + \frac{1}{y^{2n-4}x} + \ldots + \frac{1}{x^{2n-3}}\right)$. From the arithmetic-geometric mean inequality for numbers $x^{2n-1}, x^{2n-2}y, \ldots, y^{2n-1}$ we obtain $\frac{x^{2n-1}+x^{2n-2}y+\ldots+y^{2n-1}}{2n} > \sqrt[2n]{x^{2n-1} \cdot x^{2n-2}y \cdot \ldots \cdot y^{2n-1}} = \sqrt[2n]{x^{\frac{(2n-1)2n}{2}} y^{\frac{(2n-1)2n}{2}}} = (xy)^{\frac{2n-1}{2}}$. The inequality is strict, because $n > 0$ and $x \neq y$ make at least the terms x^{2n-1} and y^{2n-1} be different. For the numbers $\frac{1}{y^{2n-3}}, \frac{1}{y^{2n-4}x}, \ldots, \frac{1}{x^{2n-3}}$ we similarly obtain $\frac{\frac{1}{y^{2n-3}}+\frac{1}{y^{2n-4}x}+\ldots+\frac{1}{x^{2n-3}}}{2n-2} \geqslant \sqrt[2n-2]{\frac{1}{y^{2n-3}} \cdot \frac{1}{y^{2n-4}x} \cdot \ldots \cdot \frac{1}{x^{2n-3}}} = \sqrt[2n-2]{\frac{1}{y^{\frac{(2n-3)(2n-2)}{2}}} \cdot \frac{1}{x^{\frac{(2n-3)(2n-2)}{2}}}} = \frac{1}{(xy)^{\frac{2n-3}{2}}}$. Since $\frac{1}{y} - \frac{1}{x} = \frac{1}{xy} \cdot (x-y)$, we get $(x^{2n} - y^{2n})\left(\frac{1}{y^{2n-2}} - \frac{1}{x^{2n-2}}\right) > \frac{1}{xy} \cdot (x-y)^2 \cdot 2n(2n-2) \cdot (xy)^{\frac{2n-1}{2}} \,/\, (xy)^{\frac{2n-3}{2}} =$

$= 4n(n-1)(x-y)^2$, which proves the necessary inequality.

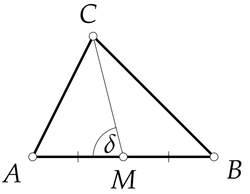

Fig. 27

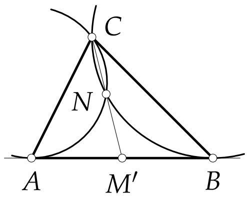

Fig. 28

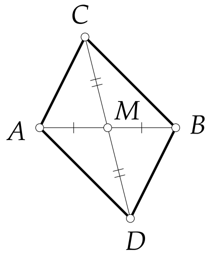

Fig. 29

S11. Let M be the midpoint of the side AB of a triangle ABC. A circle through point C that has a point of tangency to the line AB at point A and a circle through point C that has a point of tangency to the line AB at point B intersect the second time at point N. Prove that $|CM|^2 + |CN|^2 - |MN|^2 = |CA|^2 + |CB|^2 - |AB|^2$.

Solution. Denote $\angle AMC = \delta$ (Fig. 27). The law of cosines in triangles AMC and BMC gives $|CA|^2 = |CM|^2 + |AM|^2 - 2|CM||AM|\cos\delta$ and $|CB|^2 = |CM|^2 + |BM|^2 - 2|CM||BM|\cos(180^\circ - \delta)$. Observing that $|AM| = |BM|$ and adding these equations gives $|CA|^2 + |CB|^2 = 2|CM|^2 + 2|AM|^2$. Therefore $|CA|^2 + |CB|^2 - |AB|^2 = 2|CM|^2 + 2|AM|^2 - 4|AM|^2 = 2|CM|^2 - 2|AM|^2$. On the other hand, let M' be the point of intersection of lines CN and AB (Fig. 28). From the power of the point we get $|M'A|^2 = |M'C| \cdot |M'N| = |MB|^2$, therefore $|M'A| = |M'B|$, i.e., $M' = M$, which means that N is located on the line CM. So $|CN|^2 = (|CM| - |MN|)^2$ (this is true regardless of where N is located with respect to C). Therefore $|CM|^2 + |CN|^2 - |MN|^2 = |CM|^2 + (|CM| - |MN|)^2 - |MN|^2 = 2|CM|^2 - 2|CM||MN| = 2|CM|^2 - 2|AM|^2$.

Remark. The equality $|CA|^2 + |CB|^2 = 2|CM|^2 + 2|AM|^2$, which was obtained using the law of cosines, can in a more straightforward way be concluded from the so-called *parallelogram law*: the sum of the squares of the lengths of sides of the parallelogram is equal to the sum of the squares of the lengths of diagonals. Indeed, let D be the reflection of point C from point M (Fig. 29); then according to this law $|CD|^2 + |AB|^2 = 2\left(|CA|^2 + |CB|^2\right)$. By dividing both sides of the equation by two and taking into consideration that $|CD| = 2|CM|$ and $|AB| = 2|AM|$, we obtain the necessary equation.

S12. Call an n-tuple $(a_1, \ldots, a_n)$ *occasionally periodic* if there exist a non-negative integer i and a positive integer p satisfying $i + 2p \leqslant n$ and $a_{i+j} = a_{i+p+j}$ for every $j = 1, 2, \ldots, p$. Let k be a positive integer. Find the least positive integer n for which there exists an n-tuple $(a_1, \ldots, a_n)$ with elements from set $\{1, 2, \ldots, k\}$, which is not occasionally periodic but whose arbitrary extension $(a_1, \ldots, a_n, a_{n+1})$ is occasionally periodic for any $a_{n+1} \in \{1, 2, \ldots, k\}$.

Answer: $2^k - 1$.

Solution. We prove first that each tuple with the required properties contains at least $2^k - 1$ terms. Let $(a_1, \ldots, a_n)$ be such a tuple. In an arbitrary extension $(a_1, \ldots, a_n, a_{n+1})$ with $a_{n+1} \in \{1, 2, \ldots, k\}$, the end of the second instance of any repeating interval in it obviously coincides with the end of the whole $n+1$-tuple. For every $i \in \{1, 2, \ldots, k\}$, denote by p_i the length of a repeating interval in $(a_1, \ldots, a_n, i)$. Consider arbitrary two different numbers i_1, $i_2 \in \{1, 2, \ldots, k\}$. Clearly $p_{i_1} \neq p_{i_2}$, as otherwise $i_1 = a_{n+1-p_{i_1}} = a_{n+1-p_{i_2}} = i_2$. W.l.o.g., $p_{i_1} < p_{i_2}$. Suppose that $p_{i_2} < 2p_{i_1}$. Then for each positive integer $j \leqslant p_{i_2} - p_{i_1}$, $a_{n+1-2p_{i_1}+j} = a_{n+1-p_{i_1}+j} = a_{n+1-p_{i_2}+(p_{i_2}-p_{i_1})+j} = a_{n+1-2p_{i_2}+(p_{i_2}-p_{i_1})+j} = a_{n+1-2p_{i_1}-(p_{i_2}-p_{i_1})+j}$, because $j \leqslant p_{i_2} - p_{i_1} < 2p_{i_1} - p_{i_1} = p_{i_1}$, implying also $p_{i_2} - p_{i_1} + j < p_{i_2}$. So the tuple $(a_1, \ldots, a_n)$ is occasionally periodic, a contradiction. Hence $p_{i_2} \geqslant 2p_{i_1}$. As there are k lengths p_i and each two differ from each other by at least 2, the length of the longest repeating interval in a tuple $(a_1, \ldots, a_n, a_{n+1})$ is at least 2^{k-1}. Consequently, $n + 1 \geqslant 2 \cdot 2^{k-1} = 2^k$, whence $n \geqslant 2^k - 1$.

It remains to construct a tuple of length $2^k - 1$ with the required properties. For $k = 1$, we can take $L_1 = (1)$, since the only way to extend it would be with number 1 that leads to the occasionally periodic tuple $(1, 1)$. Whenever L_k is a tuple of length $2^k - 1$ with the required properties, we form a new tuple L_{k+1} that consists of all terms of L_k, followed by $k+1$, followed by all terms of L_k again. The result contains $2^{k+1} - 1$ terms that are all from the set $\{1, 2, \ldots, k+1\}$. By the choice of L_k and the construction of L_{k+1}, it is impossible to extend L_{k+1} with numbers 1, 2, $\ldots$, k. But it cannot be extended with $k+1$ either, as the extension would consist of two consecutive equal blocks.

Remark. It is natural to ask how long can a sequence containing only k different objects be if it is not occasionally periodic. It is easy to see that, in the case $k = 1$, such a sequence can contain only 1 element and, in the case $k = 2$, it can contain up to 3 elements (consecutive terms must be different and the tuple $(1, 2, 1)$ cannot be extended). Axel Thue, a Norwegian mathematician, proved more than hundred years ago that in the case $k > 2$, it is possible to construct an infinite sequence that does not contain consecutive equal blocks. This does not contradict to the statement of the problem (although there exists a tuple of length $2^k - 1$ that cannot be extended without introducing occasional periodicity, extension of many other tuples of that and greater length is possible).

Problems Listed by Topic

Number theory: O1, O6, O11, O13, F1, F5, F9, F10, F11, F14, S3, S7
Algebra: O2, O7, O12, O14, F4, F6, F8, F15, F18, S1, S5, S10
Geometry: O3, O8, O15, F2, F7, F13, F16, S4, S9, S11
Discrete mathematics: O4, O5, O9, O10, O16, F3, F12, F17, S2, S6, S8, S12