Estonian Math Competitions 2013/2014

The Gifted and Talented Development Centre

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WE THANK:

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Estonian Mathematical Olympiad

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds – at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organise olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade are allowed to participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

According to the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May. This contest lasts two days; each day, the contestants have 4.5 hours to solve 3 problems, similarly to the IMO. All participants are given the same problems. Some problems in our selection contest are at the level of difficulty of the IMO but somewhat easier problems are usually also included.

The problems of previous competitions can be downloaded from
`http://www.math.olympiaadid.ut.ee/eng`.

Besides the above-mentioned contests and the quiz "Kangaroo" some other regional competitions and matches between schools are held as well.

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This booklet contains problems that occurred in the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem to be interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

O-1. *(Juniors.)* The greatest common divisor of positive integers a, b, c is 1. It is known that c divides $a + 2b$ and $a^2 - b^2$. Prove that c also divides $a - b$.

Solution 1: Let $d = \gcd(a + b, c)$. Since $c \mid a + 2b$, also $d \mid a + 2b$. Now $(a + 2b) - (a + b) = b$ and $2(a + b) - (a + 2b) = a$ are divisible by d. Therefore d is the common divisor of a, b, c and due to our initial assumption of a, b, c being relatively prime it has to be 1. Hence $a + b$ and c are also relatively prime. But since $a^2 - b^2 = (a - b)(a + b)$ is divisible by c, the factor $a - b$ has to be divisible by c.

Solution 2: Since $a + 2b$ is divisible by c, also $(a - 2b)(a + 2b)$ is divisible by c. But $(a - 2b)(a + 2b) = a^2 - 4b^2$; since $a^2 - b^2$ is divisible by c, the differences $(a^2 - b^2) - (a^2 - 4b^2) = 3b^2$ and $4(a^2 - b^2) - (a^2 - 4b^2) = 3a^2$ also have to be divisible by c.

If $3 \nmid c$, then $c \mid a^2$ and $c \mid b^2$. Therefore every prime divisor of c would also be a prime divisor of a and b, which would contradict the initial assumption. Hence either $c = 1$, in which case the problem statement holds trivially, or $3 \mid c$. In the latter case let $c = 3c'$; the statements above show that $c' \mid a^2$ and $c' \mid b^2$. As we saw above, every prime divisor of c' would be a prime divisor of a and b, due to which $c' = 1$ and $c = 3$. Now since $a + 2b$ and $3b$ are divisible by 3, also $(a + 2b) - 3b = a - b$ is divisible by 3.

O-2. *(Juniors.)* On the board there are numbers 1, 2, 3, 4, 5 and 6. In every step Juku deletes some two numbers a and b on the board and writes $ab + a + b$ on the board instead. He repeats such steps until there is only one number on the board. Find all possibilities what could be the last number on the board.

Answer: 5039.

Solution 1: Since $(a + 1)(b + 1) = c + 1$, where c is a number that would be written on the board instead of a and b, the product of the numbers that are greater by 1 than the numbers on the board does not change in the process. In the beginning the product is $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$ or 5040. Consequently the only number that can be on the board in the end is $5040 - 1 = 5039$.

Solution 2: Let us define $x * y = xy + x + y$ for every two numbers x and y, then in each step Juku substitutes some a and b with $a * b$. The binary operation $*$ is commutative and associative, due to which the final result is independent on the order the numbers are being deleted. Therefore in the end the board will contain one number that equals $((1 * 2) * (3 * 4)) * (5 * 6) = (5 * 19) * 41 = 119 * 41 = 5039$.

O-3. *(Juniors.)* In a triangle ABC the midpoints of BC, CA and AB are D, E and F, respectively. Prove that the circumcircles of triangles AEF, BFD and CDE intersect all in one point.

Solution 1: Let us first assume that triangle ABC is not a right triangle – then the circumcenter O of the triangle ABC does not coincide with D, E, F (see fig. 1). As the circumcenter is in the point of intersection of perpendicular bisectors of the sides,

$\angle AEO = 90^\circ = \angle AFO$, due to which A, E, F, O are concyclic, so O is located on the circumcircle of AEF. Analogously O is also located on the circumcircles of BFD and CDE. Therefore O is the point we are looking for.

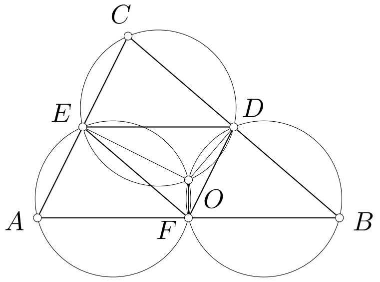

Figure 1

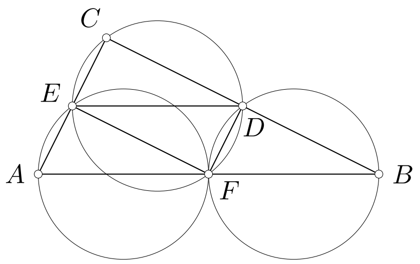

Figure 2

In the end let us also look at the case where ABC is a right triangle – without loss of generality let $\angle ACB = 90^\circ$ (see fig. 2). The circumcircles of triangles AEF and BFD obviously pass through F. As $DF \parallel AC$ and $EF \parallel BC$ by midline property, we have $DF \perp BC$ and $EF \perp AC$. Therefore also $\angle EFD = 90^\circ$. Since $\angle DCE = 90^\circ$, the line segment DE is the diameter of the circumcircle of CDE, due to which it also passes through F. Therefore F is the point we are looking for.

Solution 2: Since DE, EF and FD are the midsegments of triangle ABC, triangles AEF, FDB and ECD are congruent. Therefore their circumcircles also have radii of equal length. Let that length be r.

Let the circumcenters of AEF, BFD and CDE be G, H and I, respectively. The circumcenter of a triangle is located in the point of intersection of perpendicular bisectors of the sides, therefore G is located on the perpendicular bisector of AF and H on the perpendicular bisector of FB. As triangles AEF and FDB are congruent, points G and H are also located at equal distance from AB, due to which the distance between G and H is equal to the distance between the perpendicular bisectors of AF and FB. In conclusion

$$|GH| = \frac{1}{2}|AF| + \frac{1}{2}|FB| = \frac{1}{2}(|AF| + |FB|) = \frac{1}{2}|AB| = |AF| = |FB| = |ED| .$$

Analogously $|HI| = |BD| = |DC| = |FE|$ and $|IG| = |CE| = |EA| = |DF|$. Hence the triangle GIH is congruent to triangles AEF, FDB and ECD and the radius of the circumcircle of GIH is r. The circumcenter X of triangle GHI therefore satisfies $|XG| = |XH| = |XI| = r$, so X is located on the circumcircles of AEF, BFD and CDE.

Solution 3: A homothetic transformation with A being the homothetic center and with scaling factor $\frac{1}{2}$ takes point B to F and C to E, therefore the circumcircle of the triangle ABC goes to the circumcircle of triangle AFE. Due to factor $\frac{1}{2}$ the circumcircle of AFE passes through the circumcenter O of triangle ABC. Analogously the circumcircles of BFD and CDE also pass through O.

O-4. *(Juniors.)* A boardgame board consists of 10 squares in a row that are numbered 1 to 10. On some square there is a button. In one move it is allowed to move the button to a square whose number is either smaller by 2 or 2 times bigger. Does there exist an initial location for the button that allows the player to visit all squares of the board? It is allowed to visit one square several times.

Answer: No.

Solution: No move allowes the button to be placed to the square number 9. Therefore the button should start from there to have any hope. If on some later move the button is placed on an even-numbered square, then it will also stay on an even-numbered square on every move that follows. Therefore all the odd-numbered squares must be visited right in the beginning, i.e., the button must be moved to 7, 5, 3, 1. On the next move there is no other option but to step to square number 2. But now it is impossible to reach square number 5, since it is odd-numbered, and therefore it is also impossible to reach 10. Therefore it is not possible to visit all the squares.

O-5. *(Juniors.)* Let us call a natural number *interesting* if its any two consecutive digits form a number that is either a multiple of 19 or 21. For example the number 7638 is interesting, because 76 is a multiple of 19, 63 is a multiple of 21 and 38 is a multiple of 19. How many 2013-digit interesting numbers exist?

Answer: 9.

Solution: Amongst two-digit numbers the multiples of 19 are 19, 38, 57, 76 and 95 and the multiples of 21 are 21, 42, 63 and 84. From there we get that an interesting number cannot include digit 0 and if it happens to have digit 1, 2, 3, 4, 5, 6, 7, 8 or 9, then the next digit has to be 9, 1, 8, 2, 7, 3, 6, 4 or 5, respectively. Therefore the first digit of an interesting number also determines all the following digits. For the choice of the first digit there are 9 options, consequently there are also 9 interesting 2013-digit numbers.

O-6. *(Juniors.)* In a scalene triangle one angle is exactly two times as big as another one and some angle in this triangle is 36°. Find all possibilities, how big the angles of this triangle can be.

Answer: 18°, 36° and 126° or 36°, 48° and 96°.

Solution: Based on the initial conditions the angles of the triangle are α, 2α and $180^\circ - 3\alpha$ and they all have to be different. It remains to perform calculations for three cases: $\alpha = 36^\circ$, $2\alpha = 36^\circ$ and $180^\circ - 3\alpha = 36^\circ$.

O-7. *(Juniors.)* Let the *odd part* of a positive integer n be the greatest odd integer that divides n.

Does there exist a positive odd integer that cannot be represented as a product of the odd parts of two consecutive positive integers?

Answer: Yes.

Solution 1: Let us show that number 11 cannot be represented as a product of the odd parts of two consecutive positive integers. Assume the opposite: let $11 = x \cdot y$, where x and y are the odd parts of two consecutive positive integers. As 11 is a prime number, either $x = 1$ and $y = 11$ or $x = 11$ and $y = 1$. As of the two consecutive integers one is always odd and the odd part of an odd number is the number itself, either x or y is one of the two consecutive integers. If it is 1, then the other number can only be 2, but the odd part of 2 is not 11. If it is 11, then the other number can only be 10 or 12, but neither of those has odd part equal to 1. In all cases we got a contradiction which proves the statement.

Solution 2: Let us show 11 cannot be represented in the required way. If this representation existed, then due to the primality of 11 the factors would have to be 1 and 11. Therefore one of the two consecutive integers has to be divisible by 11. But this number cannot be 11 itself, since neither 10 nor 12 has 1 as its odd part. It also cannot be an odd multiple of 11, because then its odd part would be the number itself rather than 11. Finally, it cannot be an even multiple of 11, since in such case the neighbouring numbers would be odd numbers greater than 1, the odd parts of which are numbers themselves rather than 1.

Note: Number 11 is the least of possible examples. Indeed, let $p(n)$ stand for the odd part of number n, then $1 = 1 \cdot 1 = p(1) \cdot p(2)$, $3 = 1 \cdot 3 = p(2) \cdot p(3)$, $5 = 1 \cdot 5 = p(4) \cdot p(5)$, $7 = 7 \cdot 1 = p(7) \cdot p(8)$ and $9 = 1 \cdot 9 = p(8) \cdot p(9)$.

O-8. *(Juniors.)* a) There are three numbers a, b, c such that $a \leqslant b \leqslant c$. Let p, q, r be the pairwise sums $a+b$, $b+c$, $c+a$ in the order such that $p \leqslant q \leqslant r$. Given that $r - q = q - p$, is it certainly true that $c - b = b - a$?

b) There are four numbers e, f, g, h such that $e \leqslant f \leqslant g \leqslant h$. Let u, v, w, x, y, z be the pairwise sums of those numbers, in the order $u \leqslant v \leqslant w \leqslant x \leqslant y \leqslant z$. Given that $z - y = y - x = x - w = w - v = v - u$, is it certainly true that $h - g = g - f = f - e$?

Answer: a) Yes; b) No.

Solution: a) If $a \leqslant b \leqslant c$, then $a + b \leqslant a + c \leqslant b + c$, due to which $p = a + b$, $q = a + c$ and $r = b + c$. Equality $r - q = q - p$ can now be written as $(b + c) - (a + c) = (a + c) - (a + b)$, simplifying to $b - a = c - b$.

b) Let $e = 0$, $f = 1$, $g = 2$ and $h = 4$. Their pairwise sums in increasing order are $u = 0 + 1 = 1$, $v = 0 + 2 = 2$, $w = 1 + 2 = 3$, $x = 0 + 4 = 4$, $y = 1 + 4 = 5$ and $z = 2 + 4 = 6$. Thus $z - y = y - x = x - w = w - v = v - u = 1$, but $h - g = 2 \neq 1 = g - f$.

O-9. *(Juniors.)* In the plane there are six different points A, B, C, D, E, F such that $ABCD$ and $CDEF$ are parallelograms. What is the maximum number of those points that can be located on one circle?

Answer: 5.

Solution: As $ABCD$ and $CDEF$ are parallelograms, the line segments AB, CD and EF are parallel and have same length. Since it is impossible to draw three chords of equal length to a circle, not all 6 points can be concyclic.

A construction with 5 concyclic points is in fig. 3.

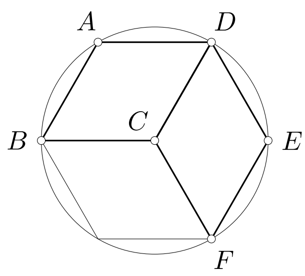

Figure 3

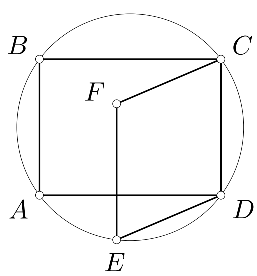

Figure 4

Note. There are many constructions with 5 vertices. We can, e.g., take a rectangle $ABCD$, add the fifth point E randomly on the circumcircle of the rectangle and choose point F such that $CDEF$ would be a parallelogram (see fig. 4).

O-10. *(Seniors.)* Find the integral part of $A = \sqrt{2013 + \sqrt{2012 + \ldots \sqrt{2 + \sqrt{1}}}}$.

Answer: 45.

Solution 1: On the one hand $A^2 > 2013 + \sqrt{2012} > 2013 + 44 > 45^2$, therefore $A > 45$. On the other hand we can demonstrate with induction that $x_n = \sqrt{n + \sqrt{n-1+\dots\sqrt{1}}} < \sqrt{n} + 1$. This holds in case of $n = 1$. Suppose it holds for some n. Then $x_{n+1} = \sqrt{n+1+x_n} < \sqrt{n+1+\sqrt{n}+1}$. It remains to show that $\sqrt{n+1+\sqrt{n}+1} < \sqrt{n+1}+1$, which is equivalent to a trivially true equation $n + \sqrt{n} + 2 < n + 2 + 2\sqrt{n+1}$. Therefore $A = x_{2013} < \sqrt{2013} + 1 < 46$.

Solution 2: Inequality $A > 45$ is proved as in solution 1. For $A < 46$, we can repeatedly use the fact that $\sqrt{x} < x$ for all $x > 1$ to obtain

$$\sqrt{2011 + \sqrt{2010 + \dots \sqrt{2 + \sqrt{1}}}} < \sqrt{2011 + 2010 + \dots + 2 + 1}$$
$$= \sqrt{\frac{2011 \cdot 2012}{2}} = \sqrt{2023066} < 1423.$$

Therefore

$$A < \sqrt{2013 + \sqrt{2012 + 1423}} = \sqrt{2013 + \sqrt{3435}} < \sqrt{2013 + 59} = \sqrt{2072} < 46.$$

Solution 3: Inequality $A > 45$ is proved as in solution 1. Notice that $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$ if $a > 0$ and $b > 0$. This gives

$$2012 + \sqrt{2011 + \dots + \sqrt{4 + \sqrt{3 + \sqrt{2 + \sqrt{1}}}}}$$
$$\leqslant 2012 + \sqrt{2011 + \dots + \sqrt{4 + \sqrt{3 + \sqrt{2} + \sqrt{\sqrt{1}}}}}$$
$$\leqslant 2012 + \sqrt{2011 + \dots + \sqrt{4 + \sqrt{3} + \sqrt{\sqrt{2}} + \sqrt{\sqrt{\sqrt{1}}}}}$$
$$\leqslant \dots \leqslant 2012 + \sqrt{2011} + \sqrt[2^2]{2010} + \dots + \sqrt[2^{2010}]{2} + \sqrt[2^{2011}]{1}$$
$$< 2012 + 45 + 7 + 3 + 2 \cdot 2008 = 6083,$$

which gives $A < \sqrt{2013 + \sqrt{6083}} < \sqrt{2013 + 78} < 46$.

Solution 4: Inequality $A > 45$ is proved as in solution 1.

Let $x_n = \sqrt{n + \sqrt{(n-1) + \dots + \sqrt{2 + \sqrt{1}}}}$, then $x_n = \sqrt{n + x_{n-1}}$. Notice that $x_n > x_{n-1}$, as in the expression of x_n, every member is greater than the corresponding member in the expression $x_{n-1} = \sqrt{(n-1) + \sqrt{(n-2) + \ldots + \sqrt{1 + \sqrt{0}}}}$. Therefore $x_{2013} > x_{2012}$ holds whence the equality $x_n^2 - x_{n-1} - n = 0$ implies $x_{2013}^2 - x_{2013} - 2013 < 0$. Solving the corresponding equation gives

$$x = \frac{1 + \sqrt{1 + 4 \cdot 2013}}{2} = \frac{1 + \sqrt{8053}}{2} < \frac{1 + \sqrt{90^2}}{2} = 45{,}5 < 46$$

for the greater root, therefore $A = x_{2013} < 46$.

Solution 5: Inequality $A > 45$ is proved as in solution 1.

Let $x_n = \sqrt{n + \sqrt{(n-1) + \ldots + \sqrt{2 + \sqrt{1}}}}$, then $x_n^2 = n + x_{n-1}$. Suppose that $x_{2013} \geqslant 46$, then $x_{2013}^2 = 2013 + x_{2012} \geqslant 46^2 = 2116$, then $x_{2012} \geqslant 103$. Analogously we would get that $x_{2011} \geqslant 8597$. On the other hand it is clear that $x_{2010} < x_{2011}$ (see solution 4), due to which $x_{2011}^2 = 2011 + x_{2010} < 2011 + x_{2011}$ and $x_{2011}(x_{2011} - 1) < 2011$. Since $x_{2011} > 2$, we get that $x_{2011} < 2011$, which contradicts the inequality $x_{2011} \geqslant 8597$. Therefore $A = x_{2013} < 46$.

O-11. *(Seniors.)* Find all natural numbers n for which there exist primes p and q such that $p(p+1) + q(q+1) = n(n+1)$.

Answer: 3 and 6.

Solution: The equation is equivalent to $p(p+1) = (n-q)(n+q+1)$. Since the difference of the factors in the r.h.s. is greater than 1, we must have $n - q < p$ and $n + q + 1 > p + 1$. As p is a prime number, $p \mid n + q + 1$. Let $n + q + 1 = kp$, $k > 1$. Now the initial equation yields $p(p+1) = (kp - 2q - 1)kp$, which is equivalent to

$$2qk = (k+1)(pk - p - 1). \tag{1}$$

As k and $k+1$ are relatively prime, $2q \mid k+1$. Since q is a prime number and $k > 1$, there are only two possibilities: $q = k+1$ or $2q = k+1$. In the first case, substituting k to (1) gives $(p-2)(q-2) = 3$, implying $p = 3$, $q = 5$ (or the other way around). Then $n = 6$ by the initial equation. In the second case similarly we get $(p-1)(q-1) = 1$, where the only possibility is $p = q = 2$ and $n = 3$.

O-12. *(Seniors.)* Find all positive real-valued solutions to

$$\begin{cases} x - y + \frac{1}{z} = 2013, \\ y - z + \frac{1}{x} = 2013, \\ z - x + \frac{1}{y} = 2013. \end{cases}$$

Answer: $x = y = z = \dfrac{1}{2013}$.

Solution 1: Suppose w.l.o.g. that $z \geqslant x$ and $z \geqslant y$. From the second equation $\dfrac{1}{x} \geqslant 2013$, therefore $x \leqslant \frac{1}{2013}$. From the third equation $\frac{1}{y} \leqslant 2013$, due to which $y \geqslant \frac{1}{2013} \geqslant x$. But now from the first equation $\frac{1}{z} \geqslant 2013$, therefore $z \leqslant \frac{1}{2013}$. Since we assumed that $z \geqslant y \geqslant \frac{1}{2013}$, the only possibility is $z = y = \frac{1}{2013}$, then also $x = \frac{1}{2013}$.

Solution 2: Adding up all the equations, we get $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3 \cdot 2013$. Multiplying the first equation by z, the second one by x and third one by y and adding together we get $3 = 2013(x + y + z)$. Therefore the arithmetic and harmonic mean of x, y, z are both equal to $\frac{1}{2013}$. Consequently $x = y = z = \frac{1}{2013}$.

O-13. *(Seniors.)* In a plane there is a triangle ABC. Line AC is tangent to circle c_A at point C and circle c_A passes through point B. Line BC is tangent to circle c_B at point C and circle c_B passes through point A. The second intersection point S of circles c_A and c_B coincides with the incenter of triangle ABC. Prove that the triangle ABC is equilateral.

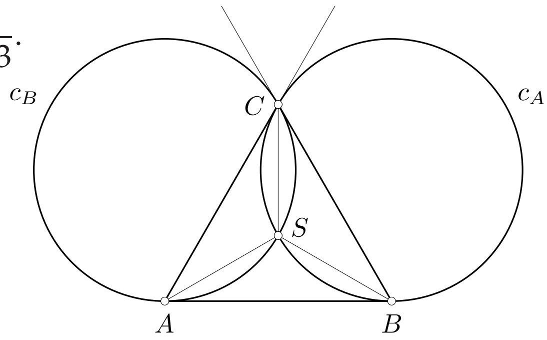

Figure 5

Solution: By the tangent-secant theorem we have $\angle BCS = \angle CAS$ and $\angle ACS = \angle CBS$ (see fig. 5). The incenter of a triangle is the point of intersection of angle bisectors, therefore $\angle CAB = 2\angle CAS = 2\angle BCS = \angle BCA$ and $\angle CBA = 2\angle CBS = 2\angle ACS = \angle BCA$. Hence ABC is equilateral.

O-14. *(Seniors.)* 20 students participated on a field trip. They all wanted to climb on top of a lighthouse, but only one person was allowed to the lighthouse at once. The order of climbing was determined by a lottery such that in the beginning every student is randomly assigned a number of 1 through 20 (such that no number is repeated). The one who gets the smallest number is the first one to climb the lighthouse. In the next round all the rest of the students are randomly assigned numbers 1 through 19 and the one who gets the smallest number gets to be the next one to climb the lighthouse. This process is repeated until all the students have climbed the lighthouse. Due to a strange occurrence no one student was assigned the same number more than once. Miku was assigned the number 14 in the first round. Find all possibilities what number could have been assigned to Miku in the 9th round.

Answer: 6.

Solution: Let the number of students be n. The last student to climb the lighthouse has got all the numbers 1 through n with the lottery. As number n is only available in the first round, that student had to get n in the first round. As number $n - 1$ is only available in 1st and 2nd round and in the 1st round that student did not get it, the student got $n - 1$ in the 2nd round. Analogously, since $n - 2$ is only available in the first three rounds and that student did not get it in the first two rounds, the student got $n - 2$ in the 3rd round. Continuing the same way shows that the last one to climb the lighthouse got numbers n through 1 in decreasing order, or got the largest available number in every round.

The rest of the students who only participated in rounds 1 through $n-1$ have to share in the first round numbers 1 through $n-1$, in the second one 1 through $n-2$, in third one 1 through $n-3$ and so on. Therefore for them this process is as if the person last to climb the tower did not partcipate at all and n would be smaller by 1. For this holds for any n, all students get the numbers in decreasing order with every next one being smaller by 1, that includes Miku. This allowes us to find that in the 9th round Miku got number 6.

O-15. *(Seniors.)* Find all pairs of positive rational numbers where the sum of the numbers in a pair is an integer and the sum of (multiplicative) inverses of the numbers in a pair is also an integer.

Answer: $(1,1)$, $(2,2)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$.

Solution 1: Let the numbers in the pair be represented as reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$. For $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ to be an integer, we must have

$$ad+bc=k\cdot bd \tag{2}$$

with k being some integer. By writing the equality (2) as $bc=(kb-a)\cdot d$ and taking into account that c and d are relatively prime, we see that b is divisible by d. By writing the same equality (2) in a form $ad=(kd-c)\cdot b$ and taking into account that a and b are relatively prime, we see that d is divisible by b. Therefore $b=d$.

For also $\frac{b}{a}+\frac{d}{c}$ to be an integer, we analogously must have $a=c$. Therefore $\frac{a}{b}=\frac{c}{d}$.

Consequently $\frac{a}{b}+\frac{c}{d}=\frac{2a}{b}$ and $\frac{b}{a}+\frac{d}{c}=\frac{2b}{a}$ are both integers. If $a=b$, then $\frac{a}{b}=1$. If $a<b$, then $\frac{2a}{b}<2$, implying $\frac{2a}{b}=1$ and $\frac{a}{b}=\frac{1}{2}$ as the only possibility. If $a>b$, then similarly $\frac{a}{b}=2$.

Solution 2: Let the numbers in the pair be represented as reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$, and let $k=\frac{a}{b}+\frac{c}{d}$ and $l=\frac{b}{a}+\frac{d}{c}$. Then $\frac{a}{b}=k-\frac{c}{d}=\frac{kd-c}{d}$ and $\frac{b}{a}=l-\frac{d}{c}=\frac{lc-d}{c}$, whence $\frac{a}{b}=\frac{c}{lc-d}$. Therefore $\frac{kd-c}{d}=\frac{c}{lc-d}$, or $cd=(kd-c)(lc-d)$. The resulting equation is equivalent to $l\cdot c^2-kld\cdot c+kd^2=0$, giving

$$c=\frac{kld\pm\sqrt{k^2l^2d^2-4kld^2}}{2l}=d\cdot\frac{kl\pm\sqrt{k^2l^2-4kl}}{2l}\,. \tag{3}$$

For c to be an integer, we must have $k^2l^2-4kl=n^2$ where n is an integer. Now

$$(kl-2)^2=n^2+4\,.$$

Therefore n^2+4 must also be a square of an integer. This is only possible when $n=0$ – therefore $kl=0$ or $kl=4$. The first option is not possible, because k and l are the sums of positive real numbers. The second option gives three possible cases: (k,l) can either be $(1,4)$, $(2,2)$ or $(4,1)$. It remains to find all possible values of $\frac{c}{d}$ from (3) and calculate $\frac{a}{b} = k - \frac{c}{d}$.

O-16. *(Seniors.)* The angles of a triangle are 22.5°, 45° and 112.5°. Prove that inside this triangle there exists a point that is located on the median through one vertex, the angle bisector through another vertex and the altitude through the third vertex.

Solution 1: Look at the triangle ABC, where $\angle CAB = 22.5^\circ$, $\angle ABC = 45^\circ$ and $\angle BCA = 112.5^\circ$.

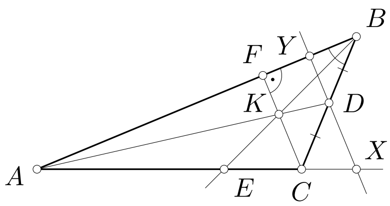

Figure 6

Let D be the point of intersection of BC and median from vertex A, E be the point of intersection of angle bisector from vertex B and AC, and F be the point of intersection of altitude from vertex C and AB (see fig. 6). As $\angle FBC = 45^\circ$ and $\angle CFB = 90^\circ$, triangle FBC is a right isosceles triangle with $|CF| = |FB|$.

Let X and Y be the points of intersection of the line that passes through point D and is parallel to CF, with lines AC and AF, respectively. We have $\angle YAX = 22.5^\circ$, $\angle XYA = 90^\circ$ and $\angle AXY = 180^\circ - \angle YAX - \angle XYA = 67.5^\circ$. Therefore

$$\angle XCD = 180^\circ - \angle BCA = 67.5^\circ = \angle AXY = \angle CXD\ ,$$

due to which $|XD| = |CD| = |DB|$.

As line segments DY and CF are parallel, DY is the midsegment of triangle BCF. Hence

$$\frac{|XD|}{|DY|} = \frac{|DB|}{|DY|} = \frac{|CB|}{|CF|} = \frac{|CB|}{|FB|}\ .$$

Let now K be the point of intersection of BE and CF. The angle bisector property gives that $\frac{|CK|}{|KF|} = \frac{|CB|}{|FB|}$, so $\frac{|XD|}{|DY|} = \frac{|CK|}{|KF|}$, from which $\angle FAK = \angle YAD$. Therefore also AD passes through K, QED.

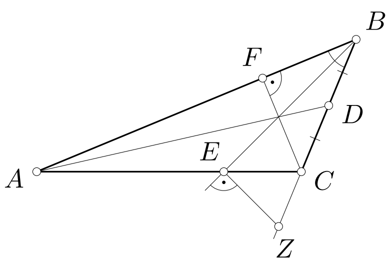

Figure 7

Solution 2: Similarly to the previous solution we pick triangle ABC, mark points D, E and F and show that $|CF| = |FB|$. Additionally notice that $\angle ABE = \frac{\angle ABC}{2} = \frac{45^\circ}{2} = 22.5^\circ = \angle BAE$, which gives $|BE| = |AE|$.

Let now Z be a point on BC such that $\angle BEZ = 90^\circ$ (see fig. 7). Then $\angle ZBE = \frac{\angle ABC}{2} = 22.5^\circ = \angle CAF$ and $\angle BEZ = 90^\circ = \angle AFC$, hence triangles BEZ and AFC are similar. Additionally

$$\angle EZC = 180^\circ - \angle ZBE - \angle BEZ = 67.5^\circ = 180^\circ - \angle BCA = \angle ECZ\ ,$$

which gives $|EZ| = |EC|$. Thus $\frac{|AE|}{|EC|} = \frac{|BE|}{|EZ|} = \frac{|AF|}{|FC|} = \frac{|AF|}{|FB|}$. Therefore $\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1$ and Ceva's theorem gives that AD, BE and CF intersect in one point.

Solution 3: Similarly to first solution we choose triangle ABC, mark points D, E, F and Y and show that $|CF| = |FB|$.

Let the points of intersection of CF with angle bisector BE and median AD be K_1 and K_2, respectively. Let $h_1 = |FK_1|$ and $h_2 = |FK_2|$ and in addition let $u = |AF|$ and $v = |CF| = |FB|$. Let us show that $h_1 = h_2$, then $K_1 = K_2$. Using the similarity of triangles BFK_1 and CFA, similarity of triangles BCK_1 and ABC and similarity of triangles AFK_2 and AYD, we get the equalities

$$\frac{h_1}{v} = \frac{v}{u}, \qquad \frac{v - h_1}{\sqrt{2}v} = \frac{\sqrt{2}v}{u+v}, \qquad \frac{h_2}{u} = \frac{\frac{1}{2}v}{u + \frac{1}{2}v}$$

which imply $\frac{v^2}{u} = h_1 = \frac{(u-v)v}{u+v}$ and $h_2 = \frac{uv}{2u+v}$. From equation $\frac{v^2}{u} = \frac{(u-v)v}{u+v}$ we get $2u + v = \frac{u^2}{v}$, therefore $h_2 = \frac{uv}{2u+v} = \frac{v^2}{u} = h_1$.

O-17. *(Seniors.)* During the schoolyear 22 olympiads were held. At each one 5 best students were awarded. It is known that the prize receivers of every two olympiads had exactly 1 student in common. Show that there exists a student who got a prize at every olympiad.

Solution: Look at an arbitrary olympiad, let that be A_1, where the prizes went to some 5 students. Each of the remaining 21 olympiads had to have someone among those 5 receiving a prize. By pigeonhole principle there exists a student who in addition to A_1 also got a prize at at least 5 olympiads. Let that student be a and those olympiads be $A_2, \ldots, A_6$.

Let now B be an arbitrary olympiad that is different from $A_1, \ldots, A_6$. As each one of the olympiads $A_1, \ldots, A_6$ has one prize-winning student in common with B and exactly 5 students get prizes at B, applying pigeonhole principle again shows that one of those five had to get a prize at at least two of $A_1, \ldots, A_6$. Since these two have student a in common and according to initial conditions that student is the only one, this means that a also got a prize at olympiad B. But since we picked B arbitrarily, a must have got a prize at every olympiad.

O-18. *(Seniors.)* a) Does there exist an integer c and a polynomial $P(x)$ with integer coefficients for which $P(c) \neq c$, but $P(P(c)) = c$?

b) Does there exist an integer c and a polynomial $P(x)$ with integer coefficients for which $P(c) \neq c$ and $P(P(c)) \neq c$, but $P(P(P(c))) = c$?

Answer: a) Yes; b) No.

Solution: a) Take $P(x) = -x$ and $c = 1$, then $P(c) = -1 \neq c$ and $P(P(c)) = P(-1) = 1 = c$.

b) Suppose that there exist a polynomial with integer coefficients $P(x)$ and an integer c, for which $P(c) \neq c$, $P(P(c)) \neq c$ and $P(P(P(c))) = c$. Notice that $P(c) \neq P(P(c))$, because otherwise $P(P(c)) = P(P(P(c))) = c$, which would contradict the assumption.

In the following calculations we will use the well-known property of polynomials with integer coefficients: $k - m \mid P(k) - P(m)$ for any distinct integers k and m.

Using this and the premise that $P(P(P(c))) = c$, we see that number $c - P(c) = P(P(P(c))) - P(P(P(P(c))))$ is divisible by $P(P(c)) - P(P(P(c))) = P(P(c)) - c$, which in turn is divisible by $P(c) - P(P(c))$, which in turn is divisible by $c - P(c)$. Hence $|c - P(c)| \geqslant |P(P(c)) - c| \geqslant |c - P(c)|$, implying $|c - P(c)| = |P(P(c)) - c|$, and similarly $|c - P(c)| = |P(c) - P(P(c))|$.

Therefore three numbers c, $P(c)$ and $P(P(c))$ are all at the same distance away from each other on the number line, which is possible only when those numbers coincide, that is $c = P(c) = P(P(c))$, which contradicts the assumptions made. This contradiction shows that polynomial $P(x)$ and integer c do not exist.

Selected Problems from the Final Round of National Olympiad

F-1. *(Grade 9.)* In the product

$$\left(1+\frac{1}{1}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{5}\right)\cdot\ldots\cdot\left(1+\frac{1}{2n-1}\right)$$

the denominators of the fractions are all odd numbers from 1 to $(2n-1)$. Is it possible to choose a natural number $n > 1$ such that this product would evaluate to an integer?

Answer: No.

Solution: By manipulating the given product we get

$$\left(1+\frac{1}{1}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{5}\right)\cdot\ldots\cdot\left(1+\frac{1}{2n-1}\right) = \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2n}{2n-1} = \frac{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}.$$

For it to be an integer, the number $2\cdot 4\cdot 6\cdot\ldots\cdot 2n$ should be divisible by $1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)$. But since

$$2\cdot 4\cdot 6\cdot\ldots\cdot 2n = (2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot\ldots\cdot(2\cdot n) = 2^n\cdot(1\cdot 2\cdot 3\cdot\ldots\cdot n)$$

and number $1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)$ is odd, number $1\cdot 2\cdot 3\cdot\ldots\cdot n$ should be divisible by $1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)$. In case of $n > 1$ this is impossible, because $1\cdot 2\cdot 3\cdot\ldots\cdot n < 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)$.

Remark: The fact that in case of $n \geqslant 2$ the fraction $\dfrac{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}$ does not evaluate to an integer can be shown more easily using Chebyshev's theorem, accorcing to which for every $n \geqslant 2$ there always exists at least one prime between n and $2n$. As this prime is certainly odd, it must occur as a factor of the product that is the denominator of the resulting fraction. But since in the numerator all the factors are in the form $2i$ where $i \leqslant n$, these factors cannot be divisible by primes greater than n.

F-2. *(Grade 9.)* A train departed from the station 12 minutes later than planned. If the train would not make any stops on the way and would travel at average speed equal to what would be the average speed between stops according to the timetable, then it would reach the destination exactly at the right time. But if the train would stop in every station for the same amount of time it was supposed to, then between the station it would have to travel with average speed 40% higher than before in order to reach the destination on time. Find the travelling time of the train according to the timetable.

Answer: 54 minutes.

Solution: Let the time we are looking for be t. The conditions of the problem imply that $t - 12\,\text{min} = 1.4 \cdot (t - 24\,\text{min})$, from which $0.4t = 21.6\,\text{min}$ and $t = 54\,\text{min}$.

F-3. *(Grade 9.)* On the sides BC, CA and AB of the triangle ABC there are points D, E and F, respectively, such that points A, B, D and E are concyclic, points B, C, E and F are concyclic and points A, C, D and F are concyclic. Does this setup require for the triangle ABC to be equilateral?

Answer: No.

Solution: Let us show that the triangle ABC can be any acute triangle. Let D, E and F be the feet of the altitudes drawn from vertices A, B and C, respectively (see fig. 8). Since $\angle ADB = 90^\circ = \angle AEB$, points D and E are on the circle with diameter AB. Analogously E and F are located on the circle with diameter BC and F and D on the circle with diameter CA.

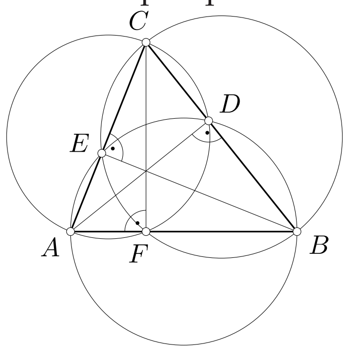

Figure 8

F-4. *(Grade 9.)* Find the greatest natural number n for which it is possible to choose n vertices of a cube such that no three of them form a right triangle.

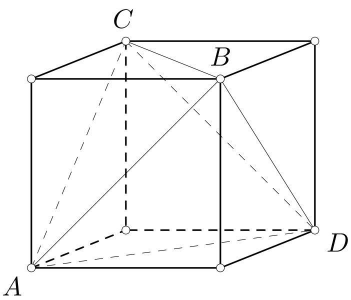

Figure 9

Answer: 4.

Solution: Let some vertex of a cube be A and let B, C and D be the opposite vertices of the faces of the cube that A belongs to (see fig. 9). Then of the vertices B, C and D any two are also the opposite vertices of some face of the cube. Therefore any two of the chosen four vertices are at the distance of a face diagonal of the cube. Therefore any three form an equilateral rather than a right triangle.

Let us now look at the situation where we choose at least 5 vertices. Two opposite faces of the cube include all the vertices of the cube. Therefore at least one of the two opposite faces has to include at least 3 of the chosen vertices. But three vertices of a square form a right triangle.

F-5. *(Grade 9.)* Juku writes down all 20-digit numbers in which each of digits 3, 4, 5 and 6 appear five times in a row (in some order). Prove that it is possible to choose two of those numbers such that their difference is divisible by 207.

Solution: As $207 = 9 \cdot 23$ and 9 and 23 are relatively prime, it suffices to find a differece that would be divisible by both 9 and 23. All the 20-digit numbers listed are divisible by 9 because the sum of their digits is 90. Therefore the difference of any two of them is also divisible by 9. It remains to show that the difference of some two of them is divisible by 23. For this note that there are 24 different orderings of 3, 4, 5 and 6. Therefore there are 24 numbers written in total, but there are 23 different possible remainders. So there must exist some two among those that give the same remainder when divided by 23. Their difference is divisible by 23.

F-6. *(Grade 10.)* Let a and n be positive integers. Prove that

$$\left\lfloor \frac{a}{n} \right\rfloor + \left\lfloor \frac{a+1}{n} \right\rfloor + \ldots + \left\lfloor \frac{a+n-1}{n} \right\rfloor = a.$$

Solution 1: Let $\left\lfloor \frac{a}{n} \right\rfloor = q$; then $a = qn + r$, where $0 \leqslant r < n$. Divide the addends given into two groups: in the first one there are $n - r$ addends, where the numerators of the fractions are equal to numbers from $qn + r$ to $qn + (n-1)$, and in the other one the rest of the r addends, where the numerators are equal to numbers from $q(n+1)$ to $q(n+1) + (r-1)$. The integral parts of the first $n - r$ fractions are equal to q and the integral parts of the last r fractions are equal to $q + 1$. Thus

$$\begin{aligned}\left\lfloor \frac{a}{n} \right\rfloor + \left\lfloor \frac{a+1}{n} \right\rfloor + \ldots + \left\lfloor \frac{a+n-1}{n} \right\rfloor &= q \cdot (n-r) + (q+1) \cdot r \\ &= qn - qr + qr + r \\ &= qn + r = a.\end{aligned}$$

Solution 2: Let us induct on a. If $a = 0$, then the numerators of the fractions are $0, 1, \ldots, n-1$, all natural numbers less than n. Thus all the integral parts in the sum are equal to 0 and the sum is 0 as required. For the inductive step the following suffices:

$$\left\lfloor \frac{a+1}{n} \right\rfloor + \ldots + \left\lfloor \frac{a+n}{n} \right\rfloor = a - \left\lfloor \frac{a}{n} \right\rfloor + \left\lfloor \frac{a}{n} + 1 \right\rfloor = a - \left\lfloor \frac{a}{n} \right\rfloor + \left\lfloor \frac{a}{n} \right\rfloor + 1 = a + 1.$$

F-7. *(Grade 10.)* Let a, b and c be real numbers for which $abc = 1$. Prove that

$$\frac{1}{1+a^{2014}} + \frac{1}{1+b^{2014}} + \frac{1}{1+c^{2014}} > 1.$$

Solution 1: Let $a^{2014} = u$, $b^{2014} = v$ and $c^{2014} = w$; then $abc = 1$ gives that $uvw = 1$. As the numerators of the l.h.s. of the inequality to be proven are positive, the inequality is equivalent to

$$(1+v)(1+w) + (1+w)(1+u) + (1+u)(1+v) > (1+u)(1+v)(1+w).$$

By expanding, simplifying and using $uvw = 1$, we get $1 + u + v + w > 0$. But this is true, since u, v and w are positive.

Solution 2: Let $a^{2014} = u$, $b^{2014} = v$ and $c^{2014} = w$. Then $abc = 1$ gives that $uvw = 1$. Therefore there exist positive real numbers x, y, z such that $u = \frac{x}{y}$, $v = \frac{y}{z}$ and $w = \frac{z}{x}$. The inequality can then be written as $\frac{1}{1+\frac{x}{y}} + \frac{1}{1+\frac{y}{z}} + \frac{1}{1+\frac{z}{x}} > 1$ which is equivalent to

$$\frac{y}{x+y} + \frac{z}{y+z} + \frac{x}{z+x} > 1.$$

But this inequality can be obtained by adding the obvious inequalities $\frac{y}{x+y} > \frac{y}{x+y+z}$, $\frac{z}{y+z} > \frac{z}{x+y+z}$ and $\frac{x}{z+x} > \frac{x}{x+y+z}$.

Solution 3: Let $a^{2014} = u$, $b^{2014} = v$ and $c^{2014} = w$; then $abc = 1$ gives that $uvw = 1$. W.l.o.g., let w be the greatest of u, v, w. Then $w \geqslant 1$, because otherwise u, v, w would all be less than 1 and their product could not be 1. Thus $uv \leqslant 1$. Now

$$\frac{1}{1+u} + \frac{1}{1+v} - 1 = \frac{(1+v) + (1+u) - (1+u)(1+v)}{(1+u)(1+v)} = \frac{1-uv}{(1+u)(1+v)} \geqslant 0.$$

Thus $\frac{1}{1+u} + \frac{1}{1+v} \geqslant 1$, from which the desired inequality can be concluded.

F-8. *(Grade 10.)* In an acute triangle ABC let the point of intersection of the altitude through B and the angle bisector through C be D. Let E be the point symmetrical to point D w.r.t. axis AC. Points A,B, C and E are concyclic. Prove that triangle ABC is isosceles.

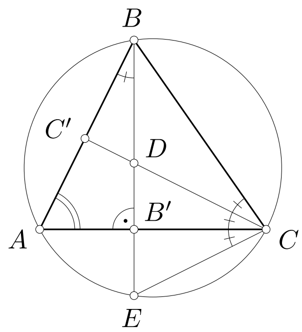

Figure 10

Solution 1: Let B' be the point of intersection of lines BD and AC and C' be the point of intersection of lines CD and AB (see fig. 10). Then $\angle ACC' = \angle ACD = \angle ACE = \angle ABE = \angle ABB'$. As triangles ABB' and ACC' share an angle at vertex A, they are similar due to having two identical angles. Therefore also $\angle AC'C = \angle AB'B = 90°$. Thus CC' is the altitude of triangle ABC, meaning that the altitude and the angle bisector through vertex C coincide. Therefore the triangle ABC is isosceles.

Solution 2: Use a known theorem according to which the orthocenter of a triangle is reflected from each side to the circumcircle of the triangle. As the point E on the extension of the altitude drawn from vertex B is located on the circumcircle of triangle ABC, point D must be the orthocenter of the triangle ABC. As the angle bisector drawn from vertex C passes through that point, it must coincide with the altitude. Therefore ABC is isosceles.

F-9. *(Grade 10.)* For which positive integers k can the integers $1, 2, 3, \ldots, (2k)^2$ be arranged as a $2k \times 2k$ table in such a way that none of the row sums and column sums had the same parity as k?

Answer: for all $k \geqslant 2$.

Solution 1: Such an arrangement is impossible for $k = 1$. In order to make all row sums and column sums in 2×2 table even, both odd numbers should occur in the same row and also in the same column, which is impossible.

1	1	1	0
1	1	0	1
1	0	0	0
0	1	0	0

Figure 11

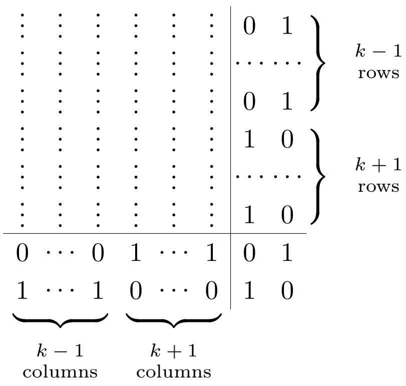

Figure 12

In the rest, let 0 and 1 denote any even and odd number, respectively. For $k = 2$, one suitable arrangement is shown in fig. 11. A way how to obtain a suitable arrangement for $k+1$ from any suitable arrangement for k is shown in fig. 12. The parity of the sum of each old row and column is inverted; each new column or row contains either k or $k+2$ odd numbers, whence the parity of the row and column sums is the opposite to that of $k+1$. Hence the extended table meets the requirements.

Solution 2: Another construction for all $k \geqslant 2$ goes as follows. For $k = 2$ and $k = 3$, suitable arrangements of 0s and 1s are depicted in fig. 13 and 14, respectively. We show now how to get an arrangement for $k+2$ from an arbitrary arrangement for k. Initially, extend the $2k \times 2k$ table with zeros by lying them as a round strip with width 2 around the table. As $(2(k+2))^2 - (2k)^2 = 16k + 16 = 16(k+1)$, exactly $8(k+1)$ of these zeros must be turned to ones.

- For odd k, the required number of zeros can be chosen blockwise with size 2×2 in an arbitrary manner. The parity of the number of ones in the existing rows and columns remain unchanged and also the new rows and columns have even sums as needed.
- If k is even then the number of ones in each row and column must be odd. By changing zeros into ones in the two blocks at both ends of one diagonal of the table, this property starts to hold. Other necessary changes can be done blockwise like in the previous case.

0	0	0	1
0	0	1	0
0	1	1	1
1	0	1	1

Figure 13

1	1	0	0	1	1
1	1	0	1	1	0
0	0	1	1	0	0
0	1	1	0	0	0
1	1	0	0	1	1
1	0	0	0	1	0

Figure 14

Solution 3: Yet another construction for all $k \geqslant 2$ follows. Divide the $2k \times 2k$ table into $2k$ cyclic diagonals (in fig. 15, one of such cyclic diagonals of the 8×8 table is coloured). Each cyclic diagonal contains exactly one cell from each row and each column.

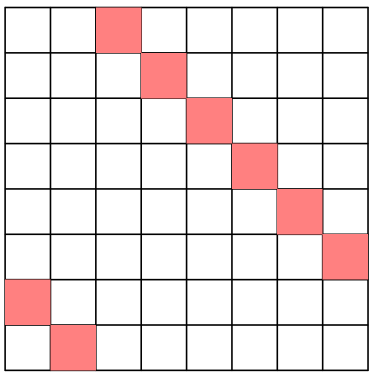

Figure 15

	0						0
1		1					
	0		0				
		1		1			
			0		0		
				1		1	
					0		0
1						1	

Figure 16

Choose two cyclic diagonals with exactly one diagonal between them. Fill both chosen diagonals with 0 and 1 alternatingly (see fig. 16).

As a result, there is either 0 or 2 ones in each row and column. Now fill half of the remaining diagonals entirely with zeros and the other half with ones. As there are $2k$ diagonals in total, exactly $k-1$ of them are filled entirely with ones. Consequently, there is either $k-1$ or $k+1$ ones in each row and column.

F-10. *(Grade 10.)* Let m be a positive integer. Prove that if Mari writes at least $m+3$ numbers on the board, then Jüri can choose 4 of those such that the sum of some two of those and the sum of the other two give the same remainder when divided by m.

Solution: As Mari writes down $m+3$ numbers and there are only m different remainders when dividing by m, there must be two that give equal remainders when divided by m; let those numbers be a and b. The rest of the $m+1$ include two that also give equal reminders when divided by m; let those be c and d. Now $a+c$ and $b+d$ give the same remainder when divided by m, thus Jüri can choose the numbers a, b, c and d.

F-11. *(Grade 11.)* How many positive integers n are there for which $2014 \cdot n$ is divisible by $2014+n$?

Solution 1: Let $d = \gcd(2014, n)$ and $2014 = da$ and $n = dx$; then a and x are relatively prime. As $2014n = d^2ax$ and $2014 + n = d(a+x)$, the number $2014n$ is divisible by $2014+n$ precisely when dax is divisible by $a+x$. As numbers a and x are relatively prime, both of them must also be relatively prime to $a+x$, which also means that ax and $a+x$ are relatively prime. Thus dax is divisible by $a+x$ precisely when d is divisible by $a+x$. This gives that $d \geqslant a$. From the equality $da = 2014$, let us look at all the cases; the positive divisors of the number 2014 are 1, 2, 19, 38, 53, 106, 1007, 2014.

- If $a = 1$, $d = 2014$, then $a + x \mid 2014$. As $a + x > 1$, there are 7 possibilities.
- If $a = 2$, $d = 1007$, then $a + x \mid 1007$. As $a + x > 2$, there are 3 possibilities.
- If $a = 19$, $d = 106$, then $a + x \mid 106$. As $a + x > 19$, there are 2 possibilities.
- If $a = 38$, $d = 53$, then $a + x \mid 53$. As $a + x > 38$, there is 1 possibility.

Those add up to 13 possibilities.

Solution 2: As the number $2014+n$ is always a divisor of $2014 \cdot (2014+n) = 2014^2 + 2014n$, the number $2014+n$ is a divisor of $2014n$ if and only if $2014+n$ is a divisor of 2014^2. The canonical representation $2014^2 = 2^2 \cdot 19^2 \cdot 53^2$ shows that the number of divisors of 2014^2 is $(2+1)\cdot(2+1)\cdot(2+1) = 27$. The middle one of them is 2014 and 13 divisors are expressible in a form $2014+n$ for some positive integer n. Thus there are 13 possibilities for n.

F-12. *(Grade 11.)* Prove that for any positive real numbers a, b and c

$$\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} > \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}.$$

Solution: By manipulating the l.h.s. we get

$$\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right). \tag{4}$$

For any real numbers x, y, z it holds that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$, because we can get it by adding together the AM-GM inequalities $\frac{1}{2}x^2 + \frac{1}{2}y^2 \geqslant xy$, $\frac{1}{2}y^2 + \frac{1}{2}z^2 \geqslant yz$ and $\frac{1}{2}z^2 + \frac{1}{2}x^2 \geqslant zx$. Thus $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$, from which

$$abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geqslant abc\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) = a + b + c.$$

Together with the equality (4) this gives

$$\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} \geqslant \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + (a+b+c). \tag{5}$$

As $\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} > x^2 + 2$ for any $x > 0$, we have

$$\left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right) + \left(c + \frac{1}{c}\right) > \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}.$$

Using this together with (5), we get the necessary inequality.

Remark. The intermediate result (5) can be proven more directly using the rearrangement inequality. Because of symmetry we can assume the ordering $a \geqslant b \geqslant c$, which implies $\frac{1}{c} \geqslant \frac{1}{b} \geqslant \frac{1}{a}$, as well as $1 + ab \geqslant 1 + ca \geqslant 1 + bc$. Thus the rearrangement inequality gives

$$\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} \geqslant \frac{1+ab}{a} + \frac{1+bc}{b} + \frac{1+ca}{c} = \frac{1}{a} + b + \frac{1}{b} + c + \frac{1}{c} + a.$$

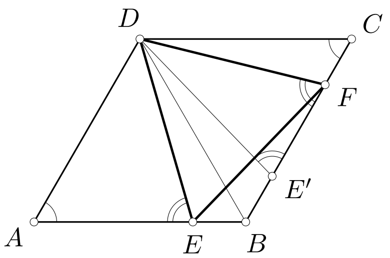

Figure 17

F-13. *(Grade 11.)* Malle drew a rhombus $ABCD$ and chose points E and F on sides AB and BC, respectively, such that the triangle DEF is equilateral. Malle was very surprised when she discovered that there is another possibility to choose points E and F on sides AB and BC, respectively, such that DEF is equilateral. What can be the measures of the angles of this rhombus?

Answer: 60° ja 120°.

Solution 1: There is clearly only one way to choose points E and F on sides AB and BC, respectively, such that E and F would be symmetrical with respect to the diagonal BD and $\angle EDF = 60^\circ$. Thus it is possible in Malle's rhombus to choose E and F asymmetrically with respect to diagonal BD such that triangle DEF is equilateral; let us consider one of the possible setups. W.l.o.g., we can assume that $|EB| < |BF|$ (otherwise we can switch the roles of A and C and the roles of E and F). Let $\alpha = \angle BAD = \angle BCD$ and $\beta = \angle AED$. Also let E' be the point symmetrical to E with respect to diagonal BD (see fig. 17); then $|DE'| = |DE| = |DF|$ gives that triangle $E'DF$

is isosceles and $\angle BFD = \angle FE'D = \angle AED = \beta$. Therefore

$$\angle CDF = 180^\circ - \angle DCF - \angle DFC = 180^\circ - \alpha - (180^\circ - \beta) = \beta - \alpha.$$

Now

$$\begin{aligned} 180^\circ - \alpha = \angle ADC &= \angle ADE + \angle EDF + \angle FDC \\ &= (180^\circ - \alpha - \beta) + 60^\circ + (\beta - \alpha) = 240^\circ - 2\alpha. \end{aligned}$$

From here $\alpha = 60^\circ$, that is the angles of the rhombus are 60° and 120°.

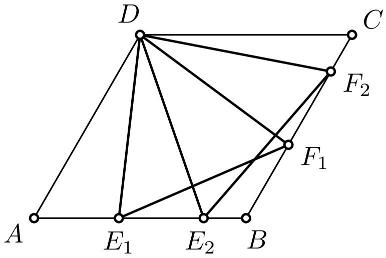

Figure 18

Solution 2: Let the first choices of Malle be E_1 and F_1 and second ones E_2 and F_2 (see fig. 18). As $\angle E_1DF_1 = 60^\circ = \angle E_2DF_2$ and $|DE_1| = |DF_1|$ and $|DE_2| = |DF_2|$, the rotation of the plane by 60° around point D that takes E_1 to F_1 must also take E_2 to F_2. The same rotation must then take the line E_1E_2 to F_1F_2 or line AB to line BC. Therefore $\angle ABC = 120^\circ$, thus the angles of the rhombus are 60° and 120°.

Remark: In a rhombus with angles 60° and 120° there are actually infinitely many possibilities to draw an equilateral triangle with one vertex in the vertex of the rhombus and the other two on the sides.

F-14. *(Grade 11.)* In a $2n \times 2n$ grid exactly half of the squares have been coloured black and the other half are white. In one step one can take some 2×2 square in this grid and reflect its four squares w.r.t. the horizontal or vertical central axis. Which positive integers n make it possible to get from any initial configuration to a state where the whole board has been coloured chessboard-style?

Answer: Every $n \geqslant 2$.

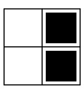
Figure 19

Solution: In case of $n = 1$ it is not possible to get the chessboard-pattern if 2×2 the initial configuration is like in fig. 19, because adjacent same-coloured squares are same-coloured also after reflecting.

Let us now show that for any $n \geqslant 2$ we can start from any initial configuration and reach the chessboard pattern. For that we can show that whenever we have some wrong-coloured squares, we can reduce their number by taking some finite number of steps. Note that a wrong-coloured square turns into a right-coloured square on the other side of the axis of reflection and the other way round. Let us define a *double reflection* to be reflecting the same 2×2 area first horizontally and then vertically. A double reflection is equivalent to a reflection with respect to the centre of the 2×2 square, whereas the wrong-coloured squares will remain wrong-coloured and the right-coloured squares right-coloured after the reflection.

First suppose that there exist two adjacent same-coloured squares. W.l.o.g., let those two wrong-coloured squares be in the same row. As $n \geqslant 2$, we can also assume that this row is at least third from the top and that there is at least one column to the right of the squares under consideration. Let us mark the wrong-coloured square with W and the right-coloured square with R on the figure; x means one or the other and x' means the opposite of x (if x is right then x' is wrong and the other way round).

- If out of the two adjacent wrong-coloured squares at least one has a wrong-coloured upper neighbour, then by reflecting w.r.t. the vertical axis the number of wrong-coloured squares decreases by at least 2 (fig. 20).

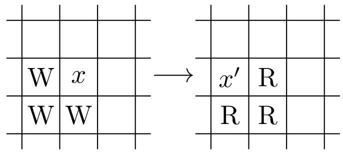

Figure 20

- If both upper neighbours of the two adjacent wrong-coloured squares are right-coloured, but at least one of them in turn has wrong-coloured upper neighbour, then by reflecting w.r.t. the vertical axis we can take the two adjacent wrong-coloured squares up by one row, so that the number of wrong-coloured squares does not change (see fig. 21). Afterwards we can do as described previously.

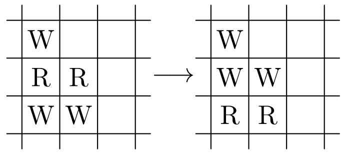

Figure 21

- If the 2×2 square above the two wrong-coloured squares is entirely right-coloured, but at least one adjacent square to the right of this 2×2 area is wrong-coloured, then we can use double reflection to swap this wrong-coloured square with one of the right-coloured squares in the 2×2 area (on fig. 22 the wrong-coloured square is the bottom right square; in the other case the same transition helps). After that we can proceed as before.

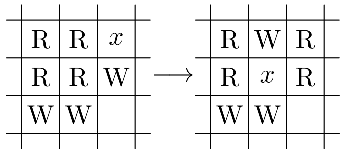

Figure 22

- In the rest of the cases we can reduce the number of wrong-coloured squares by 2 using the steps on fig. 23.

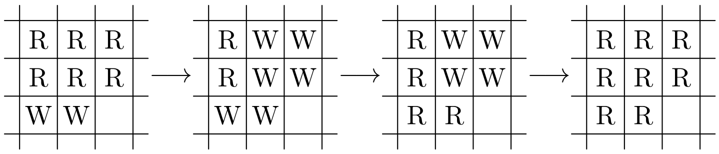

Figure 23

Let us finally look at the situation where there are no two wrong-coloured adjacent squares. With double refections a wrong-coloured square can be moved along the diagonals without changing the number of wrong- or right-coloured squares. Since the numbers of black and white squares are initially equal and do not change with reflecting, the existence of a wrong-coloured black square implies that there must also exist a wrong-coloured white square and the other way round. Therefore by taking steps along the diagonals we can take one wrong-coloured square next to another one and proceed as described above.

F-15. *(Grade 12.)* Ats and Pets both thought of two positive integers that do not exceed some positive integer n. If they both added the numbers they thought of, then both sums gave the same remainder when divided by n. But if both of them multiplied the numbers they thought of, then both products also gave equal reminders when divided by n. Is is necessarily true that the numbers they thought of were the same, if a) $n = 99$? b) $n = 101$?

Answer: a) No; b) Yes.

Solution: a) If Ats thought of numbers 1 and 21 and Pets of numbers 10 and 12, then both get the sum 22 and the products will be 21 and 120, respectively, both of which give the remainder 21 when divided by 99.

b) Let the numbers Ats chose be a and b the ones Pets chose c and d. According to the conditions stated in the problem the numbers $(a+b)-(c+d)$ and $ab-cd$ are both divisible by 101. Let $(a+b)-(c+d)=101k$; then $a=101k-b+c+d$, from where

$$\begin{aligned} ab-cd &= (101k-b+c+d)b-cd = 101kb-b^2+bc+bd-cd \\ &= 101kb-(c-b)(d-b). \end{aligned}$$

Hence also the product $(c-b)(d-b)$ is divisible by 101. As 101 is a prime number, it has to divide either the factor $c-b$ or the factor $d-b$. W.l.o.g., let $c-b$ be divisible by 101. As all the numbers are on the interval from 1 to 101, it means that $c=b$. But then $(a+b)-(c+d)=a-d$, which due to divisibility by 101 means that $a=d$. Therefore Ats and Pets must have chosen the same numbers.

Remark: In part a) there are a lot of other possibilities to show that the answer is no; e.g., Ats could have chosen 99 and 36 and Pets 33 and 3, or Ats 99 and 20 and Pets 11 and 9.

F-16. *(Grade 12.)* Find all pairs of real numbers (x,y) that satisfy

$$\begin{cases} x+\sin x &= y, \\ y+\sin y &= x. \end{cases}$$

Answer: $(k\pi,k\pi)$, where k is any integer.

Solution 1: By adding the equations and simplifying we get $\sin x=-\sin y$. Thus $y=-x+2k\pi$ or $y=\pi+x+2k\pi=x+(2k+1)\pi$. In the second case we get that $|y-x|=|(2k+1)\pi|\geqslant\pi$, but from the first equality $|y-x|=|\sin x|\leqslant 1<\pi$, a contradiction. Therefore $y=-x+2k\pi$, where k is an integer. By plugging this into the first equation we get $2x+\sin x=2k\pi$ after simplifying. We see that this is satisfied in case of any integer k by the value $x=k\pi$; then also $y=-k\pi+2k\pi=k\pi$. As $f(x)=2x+\sin x$ is an increasing function, there cannot be any other solutions to $2x+\sin x=2k\pi$.

Solution 2: Function $f(z)=z+\sin z$ is strictly increasing, because its derivative $f'(z)=1+\cos z$ is positive everywhere, except for some isolated points. Therefore if $x<y$ in case of some solution (x,y) to the system of equations, then $y=x+\sin x<y+\sin y=x$, a contradiction. Analogously $y<x$ gives a contradiction. In conclusion $x=y$ is the only option. By substituting it in we get $\sin x=\sin y=0$, from where $x=y=k\pi$ for any integer k. All pairs $(k\pi,k\pi)$ indeed satisfy the given system of equations.

F-17. *(Grade 12.)* Let I be the incenter of triangle ABC. Let R_A, R_B and R_C be the radii of the circumcircles of triangles BIC, CIA and AIB, respectively, and R be the radius of the circumcircle of triangle ABC. Prove that $R_A+R_B+R_C\leqslant 3R$.

Solution: Let $|BC|=a$, $|CA|=b$ and $|AB|=c$ and the angles opposite to those sides be α, β and γ, respectively (see fig. 24).

The law of sines in triangles ABC and IBC gives $\dfrac{a}{\sin\alpha}=2R$ and $\dfrac{a}{\sin\left(\frac{\beta}{2}+\frac{\gamma}{2}\right)}=2R_A$,

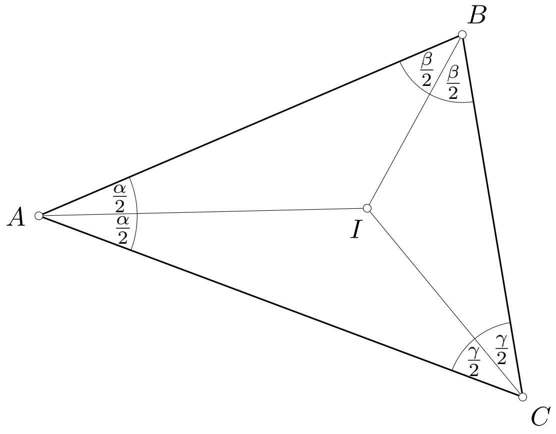

Figure 24

respectively. As $\sin\left(\frac{\beta}{2}+\frac{\gamma}{2}\right) = \sin\left(90^\circ - \frac{\alpha}{2}\right) = \cos\frac{\alpha}{2}$, we obtain $\frac{R_A}{R} = \frac{\sin\alpha}{\cos\frac{\alpha}{2}} = 2\sin\frac{\alpha}{2}$. Similarly we get $\frac{R_B}{R} = 2\sin\frac{\beta}{2}$ and $\frac{R_C}{R} = 2\sin\frac{\gamma}{2}$.

To solve the problem we need to prove that $\frac{R_A}{R}+\frac{R_B}{R}+\frac{R_C}{R} \leqslant 3$, or equivalently,

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2} \leqslant \frac{3}{2}. \tag{6}$$

Applying Jensen's inequality gives

$$\frac{1}{3}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right) \leqslant \sin\left(\frac{\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}}{3}\right) = \sin\left(\frac{\alpha+\beta+\gamma}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2},$$

which directly implies the necessary result.

F-18. *(Grade 12.)* A positive integer n is written on the board once, then $n-1$ is written on the board twice etc., on every step the number smaller by 1 from the previous number is written twice as many times as the previous number. When reaching zeros this process stops. Prove that in the end the sum of the numbers on the board is less than 2^{n+1}.

Solution: The sum of the numbers on the board is

$$s_n = 1\cdot n + 2\cdot(n-1) + 4\cdot(n-2) + \ldots + 2^{n-1}\cdot 1\,.$$

Let us also define

$$r_n = \frac{s_n}{2^n} = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \ldots + \frac{1}{2^n}\cdot n\,.$$

Notice that $r_{n+1} = \frac{r_n}{2} + \left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n+1}}\right)$ for every $n \geqslant 1$, whereby obviously

$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n+1}} < \frac{1}{2} + \frac{1}{4} + \ldots = 1$. Thus $r_n < 2$ always implies $r_{n+1} < \frac{r_n}{2} + 1 < 1 + 1 = 2$. As $r_1 = \frac{1}{2} < 2$, we have $r_n < 2$ for every $n \geqslant 1$. From there we get that $s_n = 2^n \cdot r_n < 2^{n+1}$ for every $n \geqslant 1$.

Remark: The problem can be solved also by showing $s_n = 2^{n+1} - (n+2)$ by induction.

F-19. *(Grade 12.)* Find all pairs of positive integers (x, y) for which

$$x(x+1) = y(y+1)(y^2+1).$$

Answer: $(5, 2)$.

Solution 1: Let us look at the given condition as a quadratic equation w.r.t. x. The discriminant of that is $D = 1 + 4y(y+1)(y^2+1) = 4y^4 + 4y^3 + 4y^2 + 4y + 1$. If the solution to this is an integer, then D must be a square of an integer. Notice that $(2y^2+y)^2 = 4y^4 + 4y^3 + y^2 < D$ and $(2y^2+y+1)^2 = D + y^2 - 2y$. Therefore if $y^2 - 2y > 0$, then D cannot be a square of an integer, because it is between two consecutive squares. In case of equality $y^2 - 2y = 0$ we get $y = 2$ as the only possibility, and accordingly $x = \frac{-1+\sqrt{D}}{2} = \frac{-1+(2y^2+y+1)}{2} = 5$. The case $y^2 - 2y < 0$ gives $y = 1$ as the only possibility, in which case x is not an integer.

Solution 2: The equation in the problem statement is equivalent to

$$x(x+1) = (y^2+y)(y^2+1). \tag{7}$$

If $x \leqslant y^2$, then $x(x+1) \leqslant y^2(y^2+1) < (y^2+y)(y^2+1)$, due to which the equality (7) cannot hold. If $x \geqslant y^2+y$, then $x(x+1) \geqslant (y^2+y)(y^2+y+1) > (y^2+y)(y^2+1)$ and the equality (7) cannot hold either. Therefore $x = y^2 + a$, where $0 < a < y$. By making this substitution in the equation (7), we get $(y^2+a)(y^2+a+1) = (y^2+y)(y^2+1)$, which after expanding and simplifying gives $2ay^2 + a^2 + a = y^3 + y$. This is equivalent to

$$2a(y^2+1) + a^2 - a = y(y^2+1), \tag{8}$$

from which we get that y^2+1 is a divisor of $a^2 - a$. But at the same time $a^2 - a < a^2 < y^2 < y^2 + 1$, thus $a^2 - a = 0$, giving $a = 1$ as the only possibility. By substituting it into (8), we get $2(y^2+1) = y(y^2+1)$ and $y = 2$. From here also $x = 5$.

IMO Team Selection Contest

First day

S-1. In Wonderland, the government of each country consists of exactly a men and b women, where a and b are fixed natural numbers and $b > 1$. For improving of relationships between countries, all possible working groups consisting of exactly one government member from each country, at least n among whom are women, are formed (where n is a fixed non-negative integer). The same person may belong to many working groups. Find all possibilities how many countries can be in Wonderland, given that the number of all working groups is prime.

Answer: 1.

Solution: Let r be the number of countries in Wonderland. If the minimal number of women in working groups is $n = 0$ then forming a working group means just choosing one government member from each country. Thus there are $(a+b)^r$ different working groups. This number can be prime only if $r = 1$ because $a + b \geqslant b > 1$.

If the minimal number of women in working groups is $n \geqslant 1$ then a working group containg exactly k women ($n \leqslant k \leqslant r$) can be formed as follows. Choose k countries out of r, that send a woman to that particular working group, then choose one woman out of b from each of the k governments, and finally choose one man out of a from each of the remaining $r - k$ countries. Hence there are $\binom{r}{k} b^k a^{r-k}$ working groups with exactly k women, and $\sum_{k=n}^{r} \binom{r}{k} b^k a^{r-k}$ working groups with at least n women altogether. As $n \geqslant 1$, all terms of this sum are divisible by b, whence the sum can be a prime only if it is equal to b. This is possible only if $r = 1$ since otherwise the last term (corresponding to $k = r$) of the sum would be greater than b.

The value $r = 1$ is indeed possible: if, for instance, each government consists of just 2 women then the number of all "working groups" is 2 which is a prime.

Remark: One can easily avoid referring to binomial coefficients. Let $f(x,y)$ the number of possibilities of choosing exactly one government member from x countries in such a way that the obtained set contained at least y women. We show by induction on x that $b \mid f(x,y)$ whenever $y \geqslant 1$. If $x = 0$ then $y > x$, whence a set of x persons can not contain y women. Hence $f(x,y) = 0$ which is divisible by b. Assuming the claim for x countries, consider some $x + 1$ countries; let one of them be "special". All possibilities of choosing one government member from each country so that at least y of them were women can be divided to two groups: those with a woman from the "special" country, and those with a man from the "special" country. There are at least $y - 1$ women chosen from the remaining x countries in the first case and at least y women in the second case. Thus $f(x+1,y) = b \cdot f(x,y-1) + a \cdot f(x,y)$. By the induction hypothesis, $f(x,y)$ is divisible by b, so is $f(x+1,y)$.

S-2. Let a, b and c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a^2}{b^3+c^4+1} + \frac{b^2}{c^3+a^4+1} + \frac{c^2}{a^3+b^4+1} > \frac{1}{5}.$$

Solution 1: We have $a, b, c \in (0,1)$. Therefore $b^3 < b$ and $c^4 < c$, from where $b^3 + c^4 + 1 < b + c + 1 = 1 - a + 1 = 2 - a$, whence

$$\frac{a^2}{b^3+c^4+1} > \frac{a^2}{2-a} = -2 - a + \frac{4}{2-a}.$$

Analogously $\dfrac{b^2}{c^3+a^4+1} > -2-b+\dfrac{4}{2-b}$ and $\dfrac{c^2}{a^3+b^4+1} > -2-c+\dfrac{4}{2-c}$. By adding these three inequalities we get

$$\begin{aligned}\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1} &> \\ &> -6-(a+b+c)+\left(\frac{4}{2-a}+\frac{4}{2-b}+\frac{4}{2-c}\right)= \\ &= -7+4\cdot\left(\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}\right).\end{aligned}$$

As the numbers $2-a$, $2-b$ and $2-c$ are positive, the AM-HM inequality yields that

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}\geqslant 3\cdot\frac{3}{(2-a)+(2-b)+(2-c)}=\frac{9}{6-(a+b+c)}=\frac{9}{5}.$$

In conclusion

$$\begin{aligned}\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1} &> -7+4\cdot\left(\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}\right) \\ &\geqslant -7+4\cdot\frac{9}{5}=\frac{1}{5}.\end{aligned}$$

Solution 2: By applying Cauchy-Schwarz inequality to vectors (x,y,z) and $\left(\frac{a}{x},\frac{b}{y},\frac{c}{z}\right)$, where $x=\sqrt{b^3+c^4+1}$, $y=\sqrt{c^3+a^4+1}$ and $z=\sqrt{a^3+b^4+1}$, we get

$$(x^2+y^2+z^2)\cdot\left(\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}\right)\geqslant(a+b+c)^2=1.$$

Since $a,b,c\in(0,1)$, we obtain

$$\begin{aligned}x^2+y^2+z^2 &= (b^3+c^4+1)+(c^3+a^4+1)+(a^3+b^4+1) \\ &= 3+(a^3+b^3+c^3)+(a^4+b^4+c^4) \\ &< 3+(a+b+c)+(a+b+c)=5.\end{aligned}$$

In conclusion $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}+\dfrac{c^2}{z^2}>\dfrac{1}{5}$, QED.

Solution 3: If $c\geqslant\dfrac{1}{2}$, then due to the premise $a+b+c=1$, numbers a and b are in the interval $(0,\frac{1}{2}]$. Therefore $1+a^3+b^4\leqslant 1+\left(\dfrac{1}{2}\right)^3+\left(\dfrac{1}{2}\right)^4=\dfrac{19}{16}<\dfrac{5}{4}$ and

$$\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1}>\frac{c^2}{a^3+b^4+1}>\frac{4}{5}c^2\geqslant\frac{4}{5}\cdot\frac{1}{4}=\frac{1}{5}.$$

Analogously we show the necessary inequality for cases $a\geqslant\dfrac{1}{2}$ and $b\geqslant\dfrac{1}{2}$. But if all the

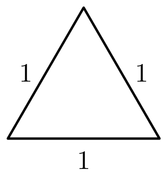

Figure 25

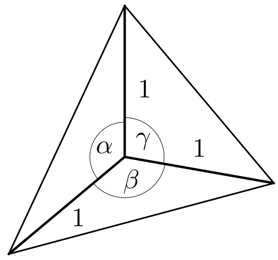

Figure 26

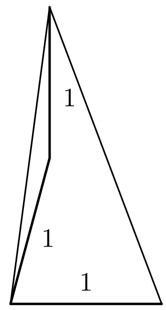

Figure 27

numbers a, b, c are less than $\frac{1}{2}$, then analogously to the previous cases the denominator of every fraction is less than $\frac{19}{16}$. Using AM-QM, we get

$$\frac{a^2}{b^3+c^4+1}+\frac{b^2}{c^3+a^4+1}+\frac{c^2}{a^3+b^4+1}>\frac{16}{19}\cdot(a^2+b^2+c^2)\geqslant\frac{16}{19}\cdot\frac{(a+b+c)^2}{3}$$
$$=\frac{16}{19\cdot 3}>\frac{1}{5}.$$

S-3. Three line segments, all of length 1, form a connected figure in the plane. Any two different line segments can intersect only at their endpoints. Find the maximum area of the convex hull of the figure.

Answer: $\frac{3}{4}\sqrt{3}$.

Solution: Clearly all vertices of the convex hull are some endpoints of the line segments. As the figure is connected, there are at most 4 different locations of the endpoints of line segments and the convex hull is either a quadrilateral or a triangle. We can assume that there are exactly 4 different locations of the endpoints of the line segments, as having only 3 meeting points would imply that the convex hull is an equilateral triangle with side length 1 whose area $S=\frac{1}{4}\sqrt{3}$ is clearly not maximal (see fig. 25).

Therefore, if the convex hull is a triangle then one of the endpoints of the line segments lies inside the triangle. We have the following three cases:

- If all line segments meet inside the triangle then the convex hull consists of three triangles, each of which has two side lengths equal to 1 (see fig. 26). Let the angles between the line segments be α, β, γ. As α, β, γ are all less than 180°, we obtain

  $$S=\frac{1}{2}(\sin\alpha+\sin\beta+\sin\gamma)\leqslant\frac{3}{2}\sin\frac{\alpha+\beta+\gamma}{3}=\frac{3}{2}\sin 120^\circ=\frac{3}{4}\sqrt{3}$$

  by Jensen's inequality. The bound $\frac{3}{4}\sqrt{3}$ is achieved when all angles between the line segments are 120°.

- If exactly two lines meet inside the triangle then the convex hull is a triangle with

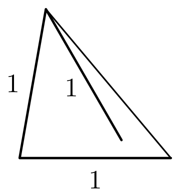

Figure 28

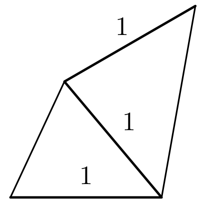

Figure 29

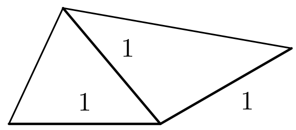

Figure 30

one side length equal to 1 and one more side length less than 2 by triangle inequality (see fig. 27). Hence $S < \frac{1}{2} \cdot 2 = 1 < \frac{3}{4}\sqrt{3}$.

- If exactly one line segment ends inside the triangle then the triangle has two sides of length 1 (see fig. 28), whence $S \leqslant \frac{1}{2} < \frac{3}{4}\sqrt{3}$.

If the convex hull is a quadrilateral then all line segments end at some vertex of the quadrilateral. We have the following cases:

- If a line segment coincides with a diagonal of the quadrilateral then other two line segments must coincide with sides of the quadrilateral (see fig. 29 and 30). So the convex hull consists of two triangles which both have two sides of length 1. Hence $S \leqslant 2 \cdot \frac{1}{2} = 1 < \frac{3}{4}\sqrt{3}$.
- If no line segment coincides with any diagonal then the line segments form 3 consecutive sides of the convex hull. Let the broken line formed by the line segments be $ABCD$. Consider two subcases:
  - If $\angle ABC + \angle BCD \leqslant 180^\circ$ (see fig. 31) then, assuming w.l.o.g. that $\angle ABC \geqslant \angle BCD$, point D lies either inside or on the boundary of the rhomboid $ABCB'$ with side length 1. Hence $S \leqslant 1 < \frac{3}{4}\sqrt{3}$.
  - If $\angle ABC + \angle BCD > 180^\circ$ then rays AB and DC meet at some point E (see fig. 32). Let $\beta = \angle EBC$, $\gamma = \angle BCE$ and $\alpha = \angle CEB$. Then $|EB| = \frac{\sin\gamma}{\sin\alpha}$ and $|EC| = \frac{\sin\beta}{\sin\alpha}$ by the law of sines in triangle EBC, and we obtain

$$\begin{aligned} S &= \frac{1}{2}\left(|EA| \cdot |ED| - |EB| \cdot |EC|\right)\sin\alpha = \frac{1}{2}\left(|EB| + |EC| + 1\right)\sin\alpha \\ &= \frac{1}{2}(\sin\alpha + \sin\beta + \sin\gamma) \leqslant \frac{3}{2}\sin\frac{\alpha+\beta+\gamma}{3} = \frac{3}{2}\sin 60^\circ = \frac{3}{4}\sqrt{3} \end{aligned}$$

    by Jensen's inequality.

Consequently, the maximum area of the convex hull is $\frac{3}{4}\sqrt{3}$.

Remark: It is possible to avoid using Jensen's inequality.

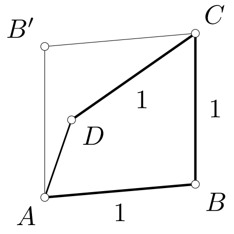

Figure 31

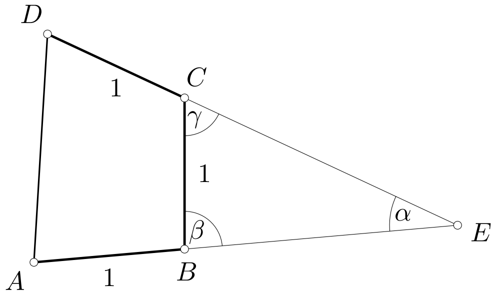

Figure 32

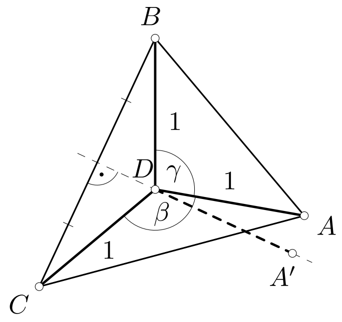

Figure 33

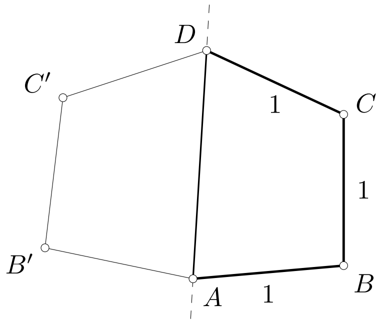

Figure 34

- Consider the case depicted in fig. 26. Let D be the common endpoint of three unit segments and let the other endpoints be A, B and C so that $\angle BDC = \alpha$, $\angle CDA = \beta$ and $\angle ADB = \gamma$. Suppose that α, β and γ are not equal; w.l.o.g., $\beta \neq \gamma$. Take a point A' on the perpendicular bisector of line segment BC on the same half plane as A from line BC in such a way that $|DA'| = 1$ (see fig. 33). As point D also lies on the perpendicular bisector of line segment BC, $\angle CDA' = \angle A'DB$ and A' lies farther away than A from BC. Consequently, interchanging DA and DA' the area of the convex hull increases. Hence the area is maximal in the case $\alpha = \beta = \gamma$. This area is $S = 3 \cdot \frac{1}{2} \sin 120^\circ = \frac{3\sqrt{3}}{4}$.
- Consider the case depicted in fig. 32. Let B' and C' be the points symmetrical to B and C, respectively, from line AD (see fig. 34). The area of quadrilateral $ABCD$ is half of the area of the hexagon $ABCDC'B'$. Note that all sides of $ABCDC'B'$ have length 1, thus the perimeter is always 6. Amongst polygons with fixed perimeter, the regular one has maximal area. Thus the area of quadrilateral $ABCD$ is maximal if the hexagon $ABCDC'B'$ is regular. This area is $S = \frac{1}{2} \cdot 6 \cdot \frac{1}{2} \sin 60^\circ = \frac{3\sqrt{3}}{4}$.

Second day

S-4. In an acute triangle the feet of altitudes drawn from vertices A and B are D and E, respectively. Let M be the midpoint of side AB. Line CM intersects the circumcircle of CDE again in point P and the circumcircle of CAB again in point Q. Prove that

$|MP| = |MQ|$.

Solution 1. The orthocenter H of the triangle ABC is located on the circumcircle of the triangle CDE, because $\angle HDC + \angle HEC = 90^\circ + 90^\circ = 180^\circ$ (see fig. 35). Let $\alpha = \angle BAC$; then also $\angle CHE = 90^\circ - \angle ECH = \alpha$. Therefore $\angle MPE = 180^\circ - \angle CPE = 180^\circ - \angle CHE = 180^\circ - \alpha$, from which we get that points A, M, P, E are concyclic. Analogously we see that points B, M, P, D are concyclic.

Point M is the circumcenter of the right triangle ABE. Therefore $|ME| = |MA|$ and $\angle MEA = \alpha$, due to which also $\angle MPA = \alpha$. But since $\angle MQB = \angle CQB = \angle CAB = \alpha$, we must have $AP \parallel BQ$. Analogously $BP \parallel AQ$.

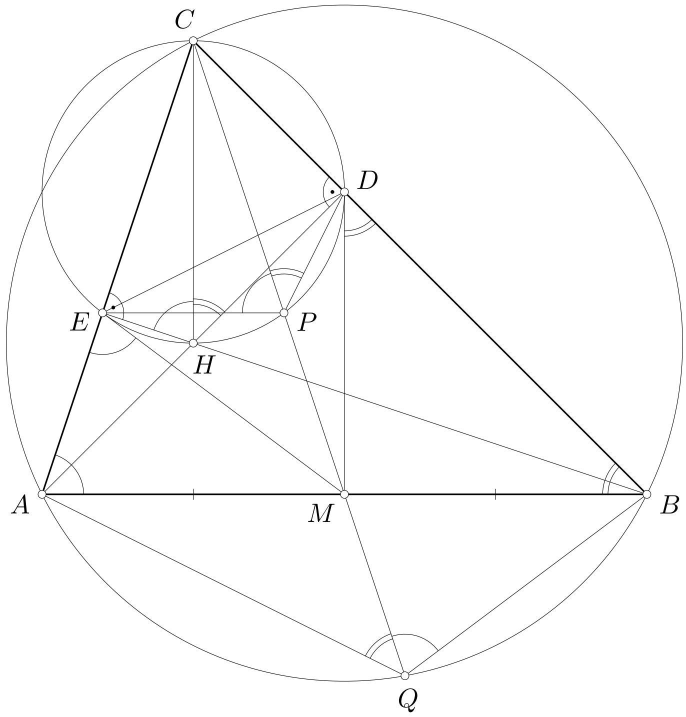

Figure 35

In conclusion we get that $APBQ$ is a parallelogram with diagonals AB and PQ. As the diagonals of a parallelogram divide each other in half, the desired claim follows.

Solution 2. Similarly to the previous solution we show that points A, M, P, E are located on one circle. Let $\vec{u} \cdot \vec{v}$ be the dot product of vectors $\vec{u}$ and $\vec{v}$. Then

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \overrightarrow{AC} \cdot \overrightarrow{EC} = \overrightarrow{MC} \cdot \overrightarrow{PC} = \overrightarrow{MC} \cdot \left(\overrightarrow{MC} - \overrightarrow{MP}\right) = \overrightarrow{MC} \cdot \overrightarrow{MC} - \overrightarrow{MC} \cdot \overrightarrow{MP}.$$

On the other hand,

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BC} &= \left(\overrightarrow{MC} - \overrightarrow{MA}\right) \cdot \left(\overrightarrow{MC} - \overrightarrow{MB}\right) = \\ &= \overrightarrow{MC} \cdot \overrightarrow{MC} - \overrightarrow{MC} \cdot \overrightarrow{MB} - \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{MA} \cdot \overrightarrow{MB} = \\ &= \overrightarrow{MC} \cdot \overrightarrow{MC} - \overrightarrow{MC} \cdot \left(\overrightarrow{MA} + \overrightarrow{MB}\right) + \overrightarrow{MA} \cdot \overrightarrow{MB}.\end{aligned}$$

In conclusion $\overrightarrow{MC} \cdot \overrightarrow{MP} = \overrightarrow{MC} \cdot \left(\overrightarrow{MA} + \overrightarrow{MB}\right) - \overrightarrow{MA} \cdot \overrightarrow{MB}$. But as M is the midpoint of AB, we have $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$, and due to choice of Q, we also have $\overrightarrow{MA} \cdot \overrightarrow{MB} = \overrightarrow{MC} \cdot \overrightarrow{MQ}$. Therefore $\overrightarrow{MC} \cdot \overrightarrow{MP} = -\overrightarrow{MC} \cdot \overrightarrow{MQ}$, meaning that

$$\overrightarrow{MC} \cdot \left(\overrightarrow{MP} + \overrightarrow{MQ}\right) = 0.$$

As $\overrightarrow{MP}$, $\overrightarrow{MQ}$ and $\overrightarrow{MC}$ have the same direction, the equality is true only if $\overrightarrow{MP} + \overrightarrow{MQ} = \vec{0}$. Therefore $|MP| = |MQ|$.

Remark. A train of thought similar to solution 2 shows that there are exactly two possibilities for choosing point M on side AB such that $|MP| = |MQ|$ would hold: CM must either be the median or altitude of ABC. Indeed, proof that A, M, P, E are on one circle does not use the premise that M is the midpoint of AB, therefore everything before applying $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$ holds true for any M on the side AB. Without substituting $\overrightarrow{MA} + \overrightarrow{MB}$ with the zero vector we eventually get

$$\overrightarrow{MC} \cdot \left((\overrightarrow{MA} + \overrightarrow{MB}) - (\overrightarrow{MP} + \overrightarrow{MQ}) \right) = 0.$$

From here $\overrightarrow{MC} \perp (\overrightarrow{MA} + \overrightarrow{MB}) - (\overrightarrow{MP} + \overrightarrow{MQ})$. Now

$$|MP| = |MQ| \iff \overrightarrow{MP} + \overrightarrow{MQ} = \vec{0} \iff \overrightarrow{MC} \perp \overrightarrow{MA} + \overrightarrow{MB}.$$

As $\overrightarrow{MA} + \overrightarrow{MB}$ has the same direction as AB, this orthogonal setup can only hold if $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$ or if MC and AB are orthogonal. Correspondingly, CM is the median or the altitude of triangle ABC.

S-5. In Wonderland there are at least 5 towns. Some towns are connected directly by roads or railways. Every town is connected to at least one other town and for any four towns there exists some direct connection between at least three pairs of towns among those four. When entering the public transportation network of this land, the traveller must insert one gold coin into a machine, which lets him use a direct connection to go to the next town. But if the traveller continues travelling from some town with the same method of transportation that took him there, and he has paid a gold coin to get to this town, then going to the next town does not cost anything, but instead the traveller gains the coin he last used back. In other cases he must pay just like when starting travelling. Prove that it is possible to get from any town to any other town by using at most 2 gold coins.

Solution 1. Let A and B be any two towns. We know that it must be possible to move from A to some other town X and from B to some other town Y. From four towns A, B, X, Y we can form three pairs which all have a direct connection between them. Of those at least one way goes from either A or X to either B or Y. Therefore it is possible to travel from A to B.

Look at some possible way of getting from A to B; let C be the first town after town A on this way and D be the last town before town B (see fig. 36). Assume that A, C, D, B are all distinct, because otherwise the problem statement follows trivially. Because of the same reason assume that there does not exist a direct connection between A and B, A and D or C and B. As according to the problem statement we can get three pairs from those four that all have direct connection between them, a direct connection must be between C and D.

Let E be some town that is not A, B, C or D. If there is a direct connection between E and A and also between E and B, then the problem statement holds. Therefore let us assume in the following that there is no direct connection between either E and A or E and B. From A, C, E and B we can form three pairs that have a direct connection between them. As there is a maximum of one direct connection between E, A and B and

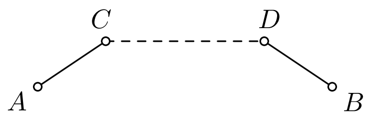

Figure 36

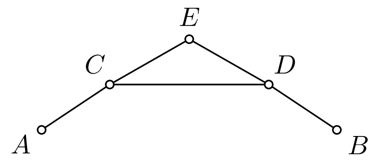

Figure 37

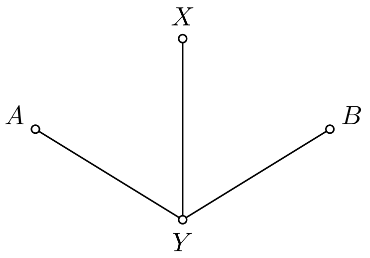

Figure 38

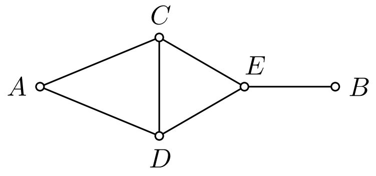

Figure 39

there is no direct connection between B and C, then there must be one between E and C. By switching the roles of A and B and also C and D we get analogously that there is a direct connection between E and D (see fig. 37).

If the connections between A and C, and D and B, are of different kind, then on the path $A \to C \to D \to B$ there must be at least two consecutive steps with same mode of transportation. For this path the problem statement holds. But if connections between A and C, and D and B, are of the same kind, then there must exist two consecutive steps with the same mode of transportation on the path $A \to C \to E \to D \to B$. For this the problem statement also holds.

Solution 2. Let A and B be any two towns. Suppose that there is no direct connection between them, because otherwise the problem statement holds trivially.

Let X be any town distinct from A and B. If there is no direct connection between A and X and no direct connection between B and X, then from a fourth town Y there must be a direct connection to A, B and X (see fig. 38). In that case one can go from A to B via Y and the problem statement holds. Because of that suppose in the following that from any town X distinct from A and B there is a direct connection to either A or B.

Let X and Y be any two distinct towns that are not A or B. Suppose that there is no direct connection between X and Y. As there is also no direct connection between A and B, but from A, B, X and Y we can form three pairs that have a direct connection between them, it is possible to go from A to B via X or Y, in which case the problem statement holds. Now the only case to look at is the one where between any two towns that are not A and B there is a direct connection.

As there are at least 5 towns in the country, there are at least 3 towns other than A and B. Therefore either A or B must have a direct connection to at least two other towns. Without loss of generality assume that A has a direct connection to C and D. But B also has a direct connection to some town E; if E coincides with any of the previously mentioned ones then the problem statement holds, which leaves us to look at the case where E is a new town. Previously mentioned facts give us that C, D and E all have direct connections between them.

If now either A and C or A and D have a direct connection between them of different kind than what is between B and E, then either path $A \to C \to E \to B$ or $A \to D \to E \to B$ has two consecutive steps with same mode of transportation. For this path the problem statement holds. But if the connection between A and C or A and D is of the same kind as the connection between B and E, then either on the path $A \to C \to D \to E \to B$ or $A \to D \to C \to E \to B$ there are two consecutive steps with same mode of transportation. For this also the problem statement holds.

S-6. Find all natural numbers n such that the equation $x^2 + y^2 + z^2 = nxyz$ has solutions in positive integers.

Answer: 1 and 3.

Solution: For $n = 1$ one of the solutions is $x = y = z = 3$ and for $n = 3$ one of the solutions is $x = y = z = 1$.

If n is even and there exists an integer solution, then the r.h.s. of the equation is even. This is possible only if at least one of the numbers x, y, z is even. Then the r.h.s. is divisible by 4. Since the remainders modulo 4 of the squares of integers can only be 0 or 1, all numbers x, y, z must be even. Let $x = 2a$, $y = 2b$, $z = 2c$. Then $a^2 + b^2 + c^2 = 2nabc$, hence (a, b, c) satisfy a similar equation with doubled n, so they must be even. Continuing this process reveals that x, y, z must be divisible by arbitrarily large power of 2 which is impossible. Consequently, there are no solutions for even n.

Suppose that for some odd $n > 3$ the equation has an integer solution (x, y, z). The given equation is equivalent to $z^2 - nxy \cdot z + (x^2 + y^2) = 0$; let z' be the other root of this quadratic equation. Then $z' > 0$ since positive nxy and $x^2 + y^2$ enable only positive solutions of the quadratic equation. By Viéte's formulae, $z' = nxy - z$. On the other hand, assume w.l.o.g. that $z = \max(x, y, z)$; then $x^2 \leqslant xz \leqslant xyz$ and $y^2 \leqslant yz \leqslant xyz$, whence $z^2 \geqslant (n-2)xyz$ and $z \geqslant (n-2)xy$. Therefore $z' \leqslant 2xy < (n-2)xy \leqslant z$, implying $x + y + z' < x + y + z$. Thus we can infinitely reduce the sum of the components of the solution, which is impossible.

Remark. Using the transformation $(x, y, z) \to (y, z, nyz - x)$, it is easy to show that in cases $n = 1$ and $n = 3$ the given equation has infinitely many solutions.

Problems Listed by Topic

Number theory: O-1, O-5, O-7, O-11, O-18, F-1, F-5, F-11, F-15, F-19, S-6.

Algebra: O-2, O-8, O-10, O-12, O-15, F-2, F-6, F-7, F-12, F-16, F-18, S-2.

Geometry: O-3, O-6, O-9, O-13, O-16, F-3, F-8, F-13, F-17, S-3, S-4.

Discrete mathematics: O-4, O-14, O-17, F-4, F-9, F-10, F-14, S-1, S-5.