Estonian Math Competitions
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Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds – at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organise olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade are allowed to participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

According to the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May. This contest lasts two days; each day, the contestants have 4.5 hours to solve 3 problems, similarly to the IMO. All participants are given the same problems. Some problems in our selection contest are at the level of difficulty of the IMO but somewhat easier problems are usually also included.

The problems of previous competitions can be downloaded from
`http://www.math.olympiaadid.ut.ee/eng`.

Besides the above-mentioned contests and the quiz "Kangaroo" some other regional competitions and matches between schools are held as well.

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This booklet contains problems that occurred in the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem to be interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

O-1. *(Juniors.)* Nonzero integers a, b and c satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. Prove that among a, b, c there are two integers which have a common divisor larger than 1.

Solution: Multiplying the given equation by abc we get $bc+ca+ab=0$. If a, b, c were all odd, then bc, ca and ab were also odd and their sum could not be 0. If one of the numbers a, b, c was even and the others were odd, then two of the numbers bc, ca and ab were even and one odd, which also would not add up to 0. Hence at least two of the numbers a, b, c are even, which satisfy the conditions.

O-2. *(Juniors.)* Teacher tells Jüri two nonzero integers a and b such that b is divisible by a. Jüri has to find a nonzero integer c such that c is divisible by b and all solutions of the quadratic equation $ax^2+bx+c=0$ are integers. Can Jüri always solve the problem?

Answer: Yes.

Solution: By the conditions of the problem there is an integer q such that $b=aq$. Let $c=-2aq^2$; then $c\neq 0$ and c is divisible by b. The quadratic equation $ax^2+bx+c=0$ or $ax^2+aqx-2aq^2=0$ has solutions q and $-2q$.

O-3. *(Juniors.)* Inside a circle c with the center O there are two circles c_1 and c_2 which go through O and are tangent to the circle c at points A and B crespectively. Prove that the circles c_1 and c_2 have a common point which lies in the segment AB.

Solution: The radius AO of the circle c is perpendicular to the common tangent to circles c and c_1 at the point A, hence AO is a diameter of the circle c_1. Similarly BO is a diameter of the circle c_2. If the circles c_1 and c_2 are tangent at the point O (Fig. 1), then the diameters AO and BO are both perpendicular to the common tangent to c_1 and c_2 at the point O, whence the lines AO and BO coincide, i.e. O lies in the segment AB. If the circles c_1 and c_2 intersect at O (Fig. 2), then let M be the other intersection point of the circles. Since $\angle AMO=90^\circ$ and $\angle BMO=90^\circ$ (angles at the circumference supported by a diameter), the lines AM and BM coincide and M lies in the segment AB.

O-4. *(Juniors.)* Numbers $1,\ldots,200$ are written on a blackboard in one line. Juku has to write in front of each number plus or minus sign so that for any positive integer $n\leq 100$ the number itself and one of its multiples have different signs. Which numbers must he assign a minus sign in order to get the maximal possible value of the expression?

Answer: The numbers $51,\ldots,100$.

Solution: If Juku writes a minus in front of the number $51,\ldots,100$ and a plus in front of the others, then the conditions of the problem are satisfied: for $51\leq n\leq 100$, the numbers n and $2n$ have different signs; for $n\leq 50$ there is at least one multiple of n among the numbers $51,\ldots,100$.

To show that this arrangement of the signs gives the maximal value of the expression,

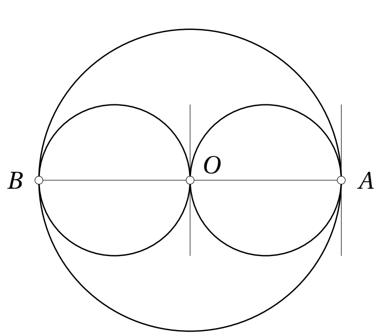

Figure 1

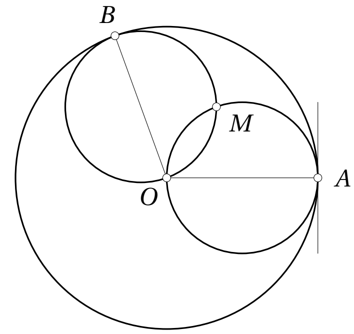

Figure 2

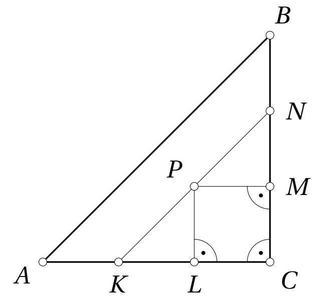

Figure 3

consider an arbitrary arrangement of signs satisfying the conditions of the problem. Then always when $100 \geq n \geq 67$ and n has a plus sign, $2n$ must have a minus sign. Also when $66 \geq n \geq 51$ and n has a plus sign, then either $2n$ or $3n$ must have a minus sign. If we change all the pluses in front of the numbers n with $100 \geq n \geq 51$ to minuses, and the minuses in front of the corresponding $2n$ or $3n$ to pluses, then changing minus to plus in front of m corresponds to changing plus to minus in front of $\frac{m}{2}$ or $\frac{m}{3}$ or both. Since $\frac{m}{2} + \frac{m}{3} < m$, the changes increase the value of the expression. Then we can also change all remaining minuses in front of the numbers $1, \ldots, 50$ and $101, \ldots, 200$ to pluses, which also can only increase the value of the expression. This results in the arrangement of the signs described in the beginning. Hence this arrangement gives the maximal value of the expression.

O-5. *(Juniors.)* Kärt writes the fractions $\frac{1}{2}$ and $\frac{1}{3}$ on the blackboard and Märt writes 10 positive integers on the paper, which he does not show to Kärt. Then Kärt starts to write fractions on the blackboard by the following rule: on each step she chooses two fractions $\frac{a}{b}$ and $\frac{c}{d}$ which are already on the blackboard and writes on the blackboard the fraction $\frac{a+c}{b+d}$ after reducing. Can Kärt always choose the fractions so that after a number of steps she writes on the blackboard a fraction whose denominator is coprime with all the numbers Märt has written on the paper?

Answer: Yes.

Solution: The first fraction that Kärt adds to the blackboard has to be $\frac{2}{5}$. On every following move, let Kärt pick $\frac{1}{2}$ as one fraction and the latest written fraction as the other fraction. Ignoring the reducing step, this means that the denominator of every added fraction is larger than the previous fraction by 2, or that the denominators of the fractions are consecutive odd numbers. This is indeed the case, because all fractions added in this way are irreducible (these fractions have the form $\frac{k}{2k+1}$, and k is always coprime with $2k+1$ because any common divisor would also divide $(2k+1) - 2k = 1$).

Therefore the denominators of the fractions that Kärt writes include all prime numbers except 2. Since there are infinitely many prime numbers, Kärt will eventually write a fraction with a prime denominator that is larger than all of the numbers written by Märt, and hence coprime with them.

O-6. *(Juniors.)* Publisher Soothsayer published a reference book claiming that for each real number x and positive even number n the equality $(1+x)^n \geq 2^n x$ holds. Is this claim true?

Answer: No.

Solution: The inequality does not hold for example when $x = \frac{1}{2}$ and $n = 4$.

Remark: This is a plausible mistake, because a similar inequality $(1+x)^n \geq 2nx$ holds for any real number x and positive even integer n.

O-7. *(Juniors.)* In an isosceles right triangle ABC the right angle is at vertex C. On the side AC points K, L and on the side BC points M, N are chosen so that they divide the corresponding side into three equal segments. Prove that there is exactly one point P inside the triangle ABC such that $\angle KPL = \angle MPN = 45^\circ$.

Solution: Without loss of generality let the points on the side AC be in the order A, K, L, C and on the side BC in the order C, M, N, B (see Fig. 3). Choose the point P so that the quadrilateral $LCMP$ is a square. Then $|KL| = |LC| = |LP|$ and $|MN| = |CM| = |MP|$, i.e. KLP and PMN are isosceles right triangles, so $\angle KPL = \angle MPN = 45^\circ$. Since $\angle KPN = 45^\circ + 90^\circ + 45^\circ = 180^\circ$, the point P lies inside the segment KN, whose all points except the endpoints are inside the triangle ABC.

To show that P is the only point with the required properties, let P' be an arbitrary point inside the triangle ABC which satisfies $\angle KP'L = \angle MP'N = 45^\circ$. Since P and P' are on the same side of the line KL and $\angle KPL = \angle KP'L$, the point P' lies on the circumcircle of the triangle KPL; similarly it also lies on the circumcircle of the triangle MPN. Since $\angle KLP = \angle PMN = 90^\circ$, the segments KP and PN are the diameters of the circles. Since the diameters KP and PN lie on the same straight line KN, they have a common perpendicular at the point P which is tangent to both circles at this point. Hence the point P is the only common point of these circles, i.e. $P' = P$.

O-8. *(Juniors.)* The numbers $1, 2, \ldots, 2012$ are written on the blackboard in some order, each of them exactly once. Between each two neighboring numbers the absolute value of their difference is written and the original numbers are erased. This is repeated until only one number is left on the blackboard. What is the largest possible number that can be left on the blackboard?

Answer: 2010.

Solution: The largest number on the blackboard cannot increase on any step, because the absolute value of the difference of two nonnegative numbers cannot be greater than the maximum of these two numbers. Since in the beginning all the numbers are different and positive, after the first step the largest possible number is 2011 and the smallest possible number is 1. After the second step the largest possible number is 2010 and hence the number left on the blackboard in the end cannot be larger than 2010.

The number 2010 can be left on the blackboard, for example when in the beginning the numbers are written in the order $2012, 1, 2, 3, \ldots, 2011$. Then after the first step there are the numbers 2011, 1, 1, ..., 1, and after the second step the numbers 2010, 0, 0, ..., 0. On each following step the number of zeroes decreases by one and in the end only the number 2010 remains.

O-9. *(Juniors.)* Find all pairs of integers (a, b) satisfying $(a+1)(b-1) = a^2b^2$.

Answer: $(0, 1)$ and $(-1, 0)$.

Solution: Since a and $a+1$ are coprime, a^2 and $a+1$ are also coprime. Similarly b^2 and $b-1$ are coprime. Hence the equality can hold only in the case $a+1 = \pm b^2$ and $b-1 = \pm a^2$, where the signs in both equations are the same.

Let both signs be pluses. Then from the first equation we get $a = b^2 - 1 = (b-1)(b+1)$. The second equation implies $b-1 = a^2$, whence $a = a^2(a^2+2)$. If $a = 0$, then $b = 1$, i.e. $(a, b) = (0, 1)$. If $a \neq 0$, then by dividing by a we get $1 = a(a^2+2)$; since $a^2+2 > 1$, this equation does not have integer solutions.

If both signs are minuses then by multiplying by -1 we get $-b+1 = a^2$ and $-a-1 = b^2$. These are the same equations with respect to $-b$ and $-a$ which we had previously with respect to a and b, hence the only solution is $-b = 0$, $-a = 1$ i.e. $(a, b) = (-1, 0)$.

O-10. *(Seniors.)* Find all positive integers which are exactly 2013 times bigger than the sum of their digits.

Answer: 36234.

Solution: Note that the minimal value of a k-digit number is 10^{k-1} and the maximal value of the cross-sum multiplied by 2013 is $9k \cdot 2013$. Since $9 \cdot 7 \cdot 2013 = 126819 < 1000000$ we can consider only numbers with up to 6 digits. Since then the cross-sum is at most 54, it is enough to consider numbers in the form $n \cdot 2013$ with $1 \leq n \leq 54$.

Since 2013 is divisible by 3, $n \cdot 2013$ and its cross-sum are divisible by 3. Since the cross-sum must be equal to n, $n \cdot 2013$ is divisible by 9. But then its cross-sum and hence also n is divisible by 9. It remains to consider the cases $n = 9, 18, \ldots, 54$ which can be checked by hand and see that only $n = 18$ satisfies the conditions.

O-11. *(Seniors.)* Find all remainders which one can get when dividing by 6 an integer n which satisfies $n^3 = m^2 + m + 1$ for some integer m.

Answer: 1.

Solution: Numbers n and n^3 give the same remainder when dividing by 6. Also, $m^2 + m + 1$ is odd and gives the remainder 0 or 1 when dividing by 3. The only possibility to get 0 as the remainder is when $m = 3k+1$, but then

$$n^3 = (9k^2 + 6k + 1) + (3k + 1) + 1 = 9k^2 + 9k + 3 = 3(3k^2 + 3k + 1)$$

which leads to a contradiction, since if n^3 is divisible by 3, it is also divisible by 3^3, but $3k^2 + 3k + 1$ is not divisible by 3. Hence the remainder of n^3 is 1 both when dividing by 2 or 3, consequently its remainder when dividing by 6 is 1.

The remainder 1 is possible: take $n = 1$ and $m = 0$ (or $n = 7$ and $m = 18$).

O-12. *(Seniors.)* Prove that for any integer $n \geq 3$ we have $(2n)! < n^{2n}$.

Solution 1: For $n = 3$ the claim holds: $(2n)! = 6! = 720$ and $n^{2n} = 3^6 = 729$.

Suppose $n \geq 4$. Divide the numbers $2, 3, \ldots, 2n-2$ into pairs $(k, 2n-k)$ with $2 \leq k \leq n-1$, leaving n alone. For each pair we have

$$k(2n-k) = (n-(n-k))(n+(n-k)) = n^2 - (n-k)^2 < n^2.$$

Hence $2 \cdot 3 \cdot \ldots \cdot (2n-2) < (n^2)^{n-2} \cdot n = n^{2n-3}$, therefore

$$(2n)! < 1 \cdot n^{2n-3} \cdot (2n-1) \cdot (2n) < n^{2n-3} \cdot (2n)^2 = 4n^{2n-1} \leq n^{2n}.$$

Solution 2: For $n = 3$ the claim holds. Suppose the claim holds for n; to show that it also holds for $n+1$ it is enough to show the inequality $(2n+1)(2n+2) < \dfrac{(n+1)^{2n}}{n^{2n}}(n+1)^2$. Since $(2n+1)(2n+2) < (2n+2)^2 = 4(n+1)^2$, it is enough to show that $\dfrac{(n+1)^{2n}}{n^{2n}} > 4$. This is equivalent with $\left(1 + \dfrac{1}{n}\right)^n > 2$ which holds for all $n \geq 2$.

O-13. *(Seniors.)* Inside a circle c there are circles c_1, c_2 and c_3 which are tangent to c at points A, B and C correspondingly, which are all different. Circles c_2 and c_3 have a common point K in the segment BC, circles c_3 and c_1 have a common point L in the segment CA, and circles c_1 and c_2 have a common point M in the segment AB. Prove that the circles c_1, c_2 and c_3 intersect in the center of the circle c.

Solution: Take a point X on the common tangent to the circles c_1 and c which lies on the other side of the line AB from the point C. Then $\angle ALM = \angle XAM = \angle XAB = \angle ACB$ (Fig. 4). Consequently $ML \parallel BC$. Similarly $KM \parallel CA$ and $LK \parallel AB$. If $\dfrac{|AM|}{|AB|} = \lambda$, then $\dfrac{|BK|}{|BC|} = \dfrac{|BM|}{|BA|} = 1 - \lambda$ and $\dfrac{|CL|}{|CA|} = \dfrac{|CK|}{|CB|} = 1 - (1-\lambda) = \lambda$, whence $\lambda = \dfrac{|AM|}{|AB|} = \dfrac{|AL|}{|AC|} = 1 - \lambda$. Hence $\lambda = \dfrac{1}{2}$, therefore the triangles AML, MBK and LKC are all similar to ABC with the factor $\dfrac{1}{2}$. Thus the radii of their circumcircles c_1, c_2 and c_3 are equal to half of the radius of the circumcircle c of the triangle ABC. Since the circles c and c_1 are tangent, the diameter of c_1 and the radius of c, both drawn from the tangent point A, coincide. Hence the circle c_1 goes through the center of the circle c; similarly the circles c_2 and c_3 go through the center of the circle c.

O-14. *(Seniors.)* For which positive integers m and n is it possible to write the numbers $1, 2, \ldots, 2mn$ into the white squares of a $2m \times 2n$ checkerboard in such a way that the sum of the numbers in every row is the same, and the sum of the numbers in every column is the same?

Answer: For all even m and n, except when $m = n = 2$.

Solution: All the numbers sum up to $mn(2mn+1)$. For odd m this is not divisible by $2n$, breaking the equality of all column sums. Thus m and likewise also n cannot be odd.

In the case $m = n = 2$ we cannot write the numbers as required, because the numbers

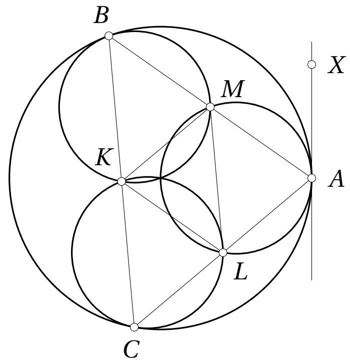

Figure 4

a		$*$	
$*$			

Figure 5

in the squares marked by $*$ in Fig. 5 must be equal.

Let us show that in the white squares of a 4×8 checkerboard we can write the numbers $k+1, k+2, \ldots, k+8$ and $2mn-k-7, 2mn-k-6, \ldots, 2mn$ so that the sums of the numbers in rows are the same, and the sums of the numbers in columns are the same. One possibility is the following, where P stands for $2mn$:

$k+1$		$P-k-2$		$k+8$		$P-k-5$	
	$k+2$		$P-k-3$		$k+7$		$P-k-4$
$P-k$		$k+3$		$P-k-7$		$k+6$	
	$P-k-1$		$k+4$		$P-k-6$		$k+5$

Here the sums in the columns are $2mn+1$ and the sums in the rows are $4mn+2$.

One can also write the numbers $1, 2, \ldots, 12$ and $2mn-11, 2mn-10, \ldots, 2mn$ on a 4×12 checkerboard in the required way, where P stands for $2mn$ again:

1		6		12		$P-3$		$P-4$		$P-9$	
	3		8		9		$P-1$		$P-6$		$P-10$
P		$P-5$		$P-11$		4		5		10	
	$P-2$		$P-7$		$P-8$		2		7		11

If one of the numbers m and n is even and the other is divisible by 4, then we can cover the $2m \times 2n$ checkerboard with 4×8 checkerboards and fill them as above, taking $k = 0, 8, \ldots, mn-8$ in different small checkerboards. If neither m nor n is divisible by 4 and one of them is at least 6 then we can cover the checkerboard with one 4×12 and 4×8 checkerboards and in the 4×8 checkerboards take $k = 12, 20, \ldots, mn-8$.

O-15. *(Seniors.)* Let a and b be positive integers such that b is divisible by a and writing a and b one after another in this order gives $(a+b)^2$. Prove that $\dfrac{b}{a} = 6$.

Solution: Let n be the number of digits of b and let $b = ka$. Then by the conditions of the problem, $10^n \cdot a + ka = (a+ka)^2$, or

$$a = \frac{10^n + k}{(k+1)^2} \,. \tag{1}$$

If k were odd, then the numerator on the r.h.s. of (1) would be odd and the denominator even, so a could not be an integer. Hence k is even.

If $k = 2$ then the cross-sum of $10^n + 2$ is 3, which is not divisible by $(2+1)^2 = 9$. The case $k = 4$ also leads to a contradiction, since $10^n + 4$ ends with 4, hence cannot be divisible by $(4+1)^2 = 25$. Thus $k \geq 6$.

In the following we show first that $k \leq 8$ and finally that $k \neq 8$. The assumptions $ka = b \geq 10^{n-1}$ give $10ka \geq 10^n$. Equality (1) implies

$$10^n = (k+1)^2 \cdot a - k = k^2 a + 2ka + a - k = (k+2) \cdot ka + a - k\,.$$

Thus $10ka \geq (k+2) \cdot ka + a - k$, whence

$$(8-k) \cdot ka \geq a - k\,. \tag{2}$$

As a is positive, $(8-k) \cdot ka > -k$. As both sides of this inequality are divisible by k, this implies $(8-k) \cdot ka \geq 0$. Consequently $8 - k \geq 0$, i.e., $k \leq 8$.

If $k = 8$, the inequality (2) implies $a \leq 8$ whereas the equality (1) reduces to $a = \dfrac{10^n + 8}{81}$. Hence a ends with digit 8, leaving $a = 8$ and $b = 8 \cdot 8 = 64$ as the only possibility. But $864 \neq (8 + 64)^2$, contradicting the conditions of the problem.

Remark: It is not hard to show that the smallest numbers satisfying the conditions of the problem are $a = \dfrac{10^{36}+6}{49} = 20408163265306122448979591836734694$ and $b = 6a = 122448979591836734693877551020408164$.

O-16. *(Seniors.)* Let x and y be different positive integers. Prove that $\dfrac{x^2+4xy+y^2}{x^3-y^3}$ is never an integer.

Solution 1: By symmetry we can assume that $x > y$. If $x - y = 1$, then

$$\begin{aligned}\frac{x^2+4xy+y^2}{x^3-y^3} &= \frac{x^2+4xy+y^2}{(x-y)(x^2+xy+y^2)} = \frac{(x-y)^2+6xy}{(x-y)\big((x-y)^2+3xy\big)} = \\ &= \frac{1+6xy}{1+3xy} = 1 + \frac{3xy}{1+3xy}\,,\end{aligned}$$

which is clearly not an integer. If $x - y \geq 2$, then

$$\begin{aligned}\frac{x^2+4xy+y^2}{x^3-y^3} &= \frac{x^2+4xy+y^2}{(x-y)(x^2+xy+y^2)} \leq \frac{x^2+4xy+y^2}{2(x^2+xy+y^2)} < \\ &< \frac{2x^2+2xy+2y^2}{2(x^2+xy+y^2)} = 1\,,\end{aligned}$$

where the last inequality follows from $x^2 - 2xy + y^2 = (x-y)^2 > 0$.

Solution 2: If $\dfrac{x^2+4xy+y^2}{x^3-y^3}$ were an integer, then

$$\frac{x^2+4xy+y^2}{x^3-y^3} \cdot (x-y) - 1 = \frac{3xy}{x^2+xy+y^2}$$

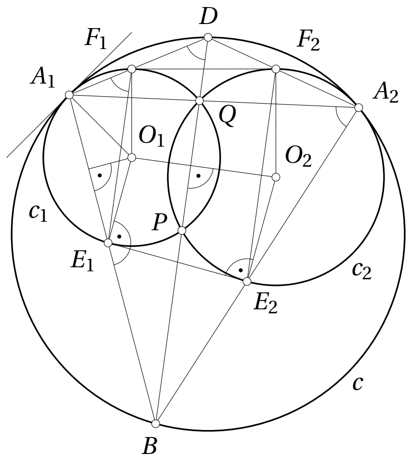

Figure 6

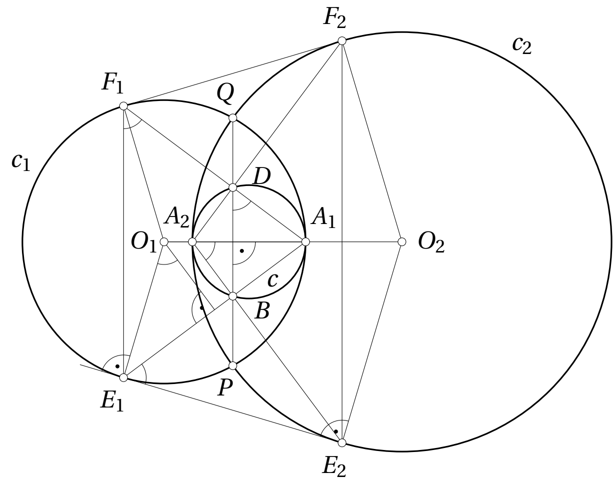

Figure 7

would also be an integer. As $3xy > 0$ and $x^2 + xy + y^2 > 0$, we have $3xy \geq x^2 + xy + y^2$, whence $(x-y)^2 \leq 0$. Hence $x = y$, which contradicts the conditions of the problem.

O-17. *(Seniors.)* Circles c_1, c_2 with centers O_1, O_2, respectively, intersect at points P and Q and touch circle c internally at points A_1 and A_2, respectively. Line PQ intersects circle c at points B and D. Lines A_1B and A_1D intersect circle c_1 the second time at points E_1 and F_1, respectively, and lines A_2B and A_2D intersect circle c_2 the second time at points E_2 and F_2, respectively. Prove that E_1, E_2, F_1, F_2 lie on a circle whose center coincides with the midpoint of line segment O_1O_2.

Solution: Let the radii of c_1, c_2 and c be r_1, r_2 and r, respectively. Homothety of ratio $\dfrac{r}{r_1}$ with center A_1 takes circle c_1 to circle c and points E_1, F_1 to points B, D, respectively. Thus it takes line E_1F_1 to line BD. Analogously, homothety of ratio $\dfrac{r}{r_2}$ with center A_2 takes line E_2F_2 to line BD. Consequently, lines E_1F_1 and E_2F_2 are parallel to line BD (Fig. 6).

Furthermore, note that $|BE_1| \cdot |BA_1| = |BP| \cdot |BQ|$ and $|BE_2| \cdot |BA_2| = |BP| \cdot |BQ|$, implying $|BE_1| \cdot |BA_1| = |BE_2| \cdot |BA_2|$. Thus triangles BE_1E_2 and BA_2A_1 are similar and

$$\angle BE_1E_2 = \angle BA_2A_1 = \angle BDA_1 = \angle E_1F_1A_1 = \frac{1}{2}\angle E_1O_1A_1 = 90^\circ - \angle O_1E_1A_1\,, \quad (3)$$

whence

$$\angle E_2E_1O_1 = 180^\circ - \angle BE_1E_2 - O_1E_1A_1 = 90^\circ\,. \quad (4)$$

Analogously, $\angle E_1E_2O_2 = 90^\circ$. Hence the quadrilateral $E_1E_2O_2O_1$ is a right-angled trapezoid (or rectangle in the case $r_1 = r_2$) and the midpoint of the line segment O_1O_2 lies on the perpendicular bisector of the line segment E_1E_2, thus being equidistant from E_1 and E_2. Analogously, the midpoint of the line segment O_1O_2 is also equidistant from F_1 and F_2.

As line O_1O_2 is perpendicular to BD, line O_1O_2 is also perpendicular to E_1F_1. Thus the line segment O_1O_2 entirely lies on the perpendicular bisector of E_1F_1. This means that the midpoint of line segment O_1O_2 is equidistant from E_1 and F_1.

Altogether, we have shown that these four points lie on a circle with its center at the midpoint of the line segment O_1O_2.

Remark: The chains of equations (3) and (4) hold as given in the situation depicted in Fig. 6, where F_1 and O_1 lie at the same side from line A_1E_1. There are other situations where F_1 and O_1 lie at different sides from A_1E_1 or O_1 lies on the line A_1E_1 or circle c lies inside circles c_1 and c_2 (see Fig. 7). Despite the equations having a little different form, the final result $\angle E_2E_1O_1 = 90^\circ$ still holds.

O-18. *(Seniors.)* Eha and Koit play the following game. In the beginning of the game at each vertex of a square there is an empty box. At each step each player has two possibilities: either add one stone to an arbitrary box, or to move each box clockwise to the next vertex of the square.

Koit begins and they make in turns 2012 steps (each player 1006). Then Koit marks one of the vertices of the square and lets Eha make one more step. Koit wins if after this step the number of stones in some box is larger than the number of stones in the box at the vertex Koit marked; otherwise Eha wins. Which player has a winning strategy?

Answer: Eha.

Solution: First show that Eha can guarantee that before Koit's move the number of stones in the boxes lying at opposite corners of the square are equal. In the beginning it is true, since all the boxes are empty. Let before Koit's move the numbers of stones in the boxes be (a, b, a, b). If Koit adds one stone to a box, then Eha can add a stone to the box at the opposite corner; if Koit moves the boxes cyclically, Eha also moves the boxes cyclically, so the condition still holds. Hence Eha can make the moves so that after step 2012 the numbers of the stones in the boxes are (a, b, a, b). Without loss of generality we can assume that $a \geq b$. If Koit marks a vertex with a box with a stones then Eha adds one stone to the box and wins. If Koit marks a vertex with a box with b stones, then Eha moves the boxes cyclically and still wins.

O-19. *(Seniors.)* Find all functions f from the set of all positive integers to the same set such that, for all positive integers $a_1, \ldots, a_k$ with $k > 0$, the sum $a_1 + \ldots + a_k$ divides the sum $f(a_1) + \ldots + f(a_k)$.

Answer: All functions given by $f(n) = an$, $a \in \mathbb{N}$.

Solution: Suppose that f is a function that satisfies the conditions of the problem. We claim that $f(n) = f(n-1) + f(1)$ for all integers $n > 1$. Indeed, for any integer $m > n$, we have $m \mid f(n) + f(m-n)$ and $m \mid f(n-1) + f(1) + f(m-n)$ by conditions of the problem. Hence the difference $f(n) - (f(n-1) + f(1))$ is also divisible by m. As m was arbitrary, this implies that $f(n) - (f(n-1) + f(1))$ is divisible by an infinite number of different integers, i.e., is equal to 0. This completes the proof of the claim.

Easy induction now gives that necessarily $f(n) = nf(1)$. It is straightforward to verify that all functions of the form $f(n) = an$ satisfy the conditions of the problem.

Selected Problems from the Final Round of National Olympiad

F-1. *(Grade 9.)* Consider hexagons whose internal angles are all equal.

(i) Prove that for any such hexagon the sum of the lengths of any two neighbouring sides is equal to the sum of the lengths of their opposite sides.

(ii) Does there exist such a hexagon with side lengths 1, 2, 3, 4, 5 and 6 in some order?

Answer: ii) Yes.

Solution 1:

(i) Let the hexagon be $ABCDEF$. It suffices to show that $|AB| + |BC| = |DE| + |EF|$.

Let K be the intersection point of rays FA and CB and let L be the intersection point of rays FE and CD (Fig. 8). The size of every internal angle of the hexagon is 120°, whence triangles KAB and LDE are equilateral. The quadrilateral $FKCL$ is a parallelogram since its opposite sides are parallel. This implies $|KC| = |LF|$ or $|KB| + |BC| = |LE| + |EF|$, which together with $|KB| = |AB|$ and $|LE| = |DE|$ implies the desired equality $|AB| + |BC| = |DE| + |EF|$.

(ii) Take a parallelogram with side lengths 7 and 5 and internal angles 60° and 120°, and cut off equilateral triangles with side lengths 1 and 2 at its acute angles. This gives rise to a hexagon with all internal angles having size 120° and side lengths 1, 4, 5, 2, 3, 6 (Fig. 9).

Solution 2:

(i) Note that the external angles of the hexagon have size 60°. Any two opposite sides of the hexagon are parallel, as they are separated by exactly three external angles.

Consider a line s perpendicular to opposite sides CD and FA of the hexagon $ABCDEF$ (Fig. 10). As all other sides form the same angle 30° with line s, the lengths of these sides are proportional to the lengths of the projections of the sides to line s. The sum of the lengths of the projections of sides AB and BC is equal to the distance between the parallel lines CD and FA and the same holds also for the opposite sides DE and EF. Therefore the sum of the lengths of the projections of sides AB and BC is equal to that of sides DE and EF, whence the sums of the lengths of the sides are equal as well.

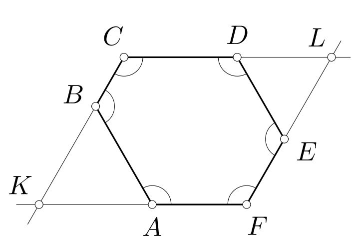

Figure 8

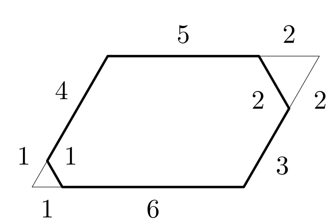

Figure 9

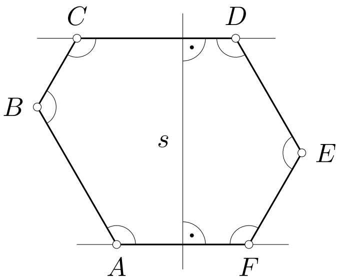

Figure 10

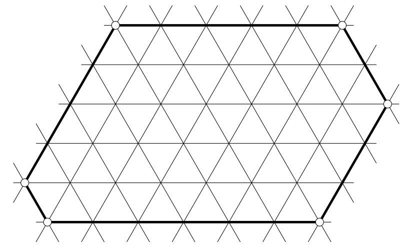
Figure 11

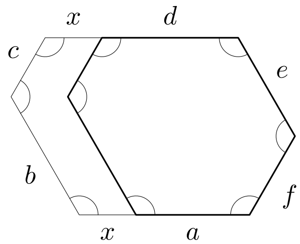

Figure 12

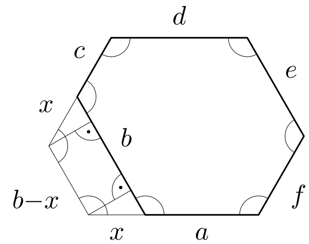

Figure 13

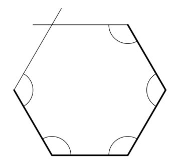
Figure 14

(ii) Figure 11 shows a hexagon in a triangular grid with distance between neighbouring nodes being 1. The side lengths of the hexagon are 1, 4, 5, 2, 3 and 6.

Solution 3: Consider two types of transformations on hexagons that maintain the property that all internal angles are of the same size.

(1) Prolonging two opposite sides by the same quantity x (Fig. 12); this causes the side lengths to change according to the template

$$(a, b, c, d, e, f) \longleftrightarrow (a+x, b, c, d+x, e, f) .$$

(2) Prolonging the two neighbouring sides of one particular side by the same quantity x (Fig. 13); this causes the side lengths to change according to the template

$$(a, b, c, d, e, f) \longleftrightarrow (a+x, b-x, c+x, d, e, f)$$

A straightforward check shows that both transformations maintain the desired property, no matter of in which direction the transformations are applied.

(i) As the internal angles of all regular hexagons are equal, it suffices to show that an arbitrary hexagon with all internal angles equal can be turned into a regular hexagon by a finite sequence of the transformations above.

Indeed, let the side lengths of a given hexagon with all internal angles equal be (a, b, c, d, e, f). Assume w.l.o.g. that $d \geq a$ and $f \geq c$. Choose a quantity s such that $s \geq \max(a, b, c)$; by applying the transformation (1) thrice, we can obtain a hexagon with three consecutive sides having the same length:

$$(a, b, c, d, e, f) \xrightarrow{(1)} (s, b, c, d', e, f) \xrightarrow{(1)} (s, s, c, d', e', f) \xrightarrow{(1)} (s, s, s, d', e', f') .$$

By the assumption made above we have $d' \geq s$ and $f' \geq s$. W.l.o.g., assume also $d' \leq f'$. The transformation

$$(s, s, s, d', e', f') \xrightarrow{(2)} (s, s, s, s, e'', f'')$$

leads to a hexagon with four consecutive sides of equal length. But this must be regular since all of its internal angles are equal (Fig. 14).

(ii) Such a hexagon can be obtained from a regular hexagon with side length 1 by the following transformations:

$$(1, 1, 1, 1, 1, 1) \xrightarrow{(1)} (1, 4, 1, 1, 4, 1) \xrightarrow{(1)} (1, 4, 5, 1, 4, 5) \xrightarrow{(2)} (1, 4, 5, 2, 3, 6) .$$

F-2. *(Grade 9.)* Two children are playing noughts and crosses with changed rules. In each move, either of the players may draw into an empty square of a 3×3 board either a nought or a cross according to one's wish. Moves are made alternately and the winner is the one after whose move a row, a column or a long diagonal becomes filled with three similar signs. Is there a player with a winning strategy, and if yes then who?
Answer: Yes, the first player.

Solution: The first player may play the first move into the middle square and, later on, make the immediately winning move if there is any and play symmetrically to the opponent's last move w.r.t. the center of the board otherwise.

Suppose that the opponent wins. As the central square is occupied, the winning move must be played either into a corner or in the middle of an edge of the board. According to the first player's strategy, the position before the winning move was symmetric w.r.t. the center of the board. Consequently, the square symmetric to the winning move is empty in the final position, which in turn implies that the three signs of the same type appear along an edge of the board. Before the winning move, there must already have been two of these signs present and, by symmetry, similarly also at the opposite edge. Three of these four signs must already have been there before the last move of the first player. As two of these three must have been in one line, the first player could win in her last move, which contradicts the chosen strategy.

Remark: It is also easy to argue by case study.

F-3. *(Grade 10.)* Can 2013 be represented as the difference of two cubes of integers?

Answer: No.

Solution 1: Suppose that $2013 = x^3 - y^3$ where x and y are integers. Note that

$$x^3 - y^3 = (x-y)^3 + 3x^2y - 3xy^2 = (x-y)^3 + 3xy(x-y) .$$

As 2013 is divisible by 3 and so is $3xy(x-y)$, the difference $(x-y)^3$ must be divisible by 3. Thus also $x - y$ is divisible by 3 as 3 is prime. Consequently, $3xy(x-y)$ is divisible by 3^2 and $(x-y)^3$ is divisible by 3^3, whence the sum $x^3 - y^3$ is divisible by 3^2. But 2013 is not divisible by higher powers of 3. The contradiction shows that 2013 cannot be represented as the difference of two cubes of integers.

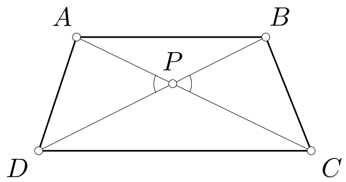

Figure 15

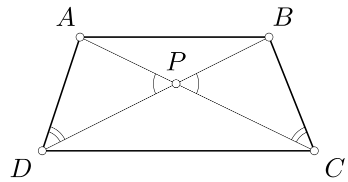

Figure 16

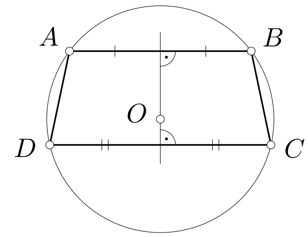

Figure 17

Solution 2: Suppose that $2013 = x^3 - y^3$ where x and y are integers. Then $x^3 \equiv y^3 \pmod{3}$. As integers are congruent to their cubes modulo 3, this gives $x \equiv y \pmod{3}$.

Now write $2013 = (x - y)(x^2 + xy + y^2)$. By the congruence obtained above, the first factor in the r.h.s. is divisible by 3 and the terms x^2, xy and y^2 in the second factor are all congruent modulo 3 whence $x^2 + xy + y^2$ is divisible by 3. Altogether, the product $(x - y)(x^2 + xy + y^2)$ must be divisible by 9, but 2013 is not.

Remark: It is possible to solve this problem by brute force in several ways.

F-4. *(Grade 10.)* The bases of trapezoid $ABCD$ are AB and CD, and the intersection point of its diagonals is P. Prove that if $\dfrac{|PA|}{|PD|} = \dfrac{|PB|}{|PC|}$ then the trapezoid is isosceles.

Solution 1: By assumptions, $\dfrac{|PA|}{|PB|} = \dfrac{|PD|}{|PC|}$. As the bases AB and CD are parallel, we have also $\dfrac{|PA|}{|PB|} = \dfrac{|PC|}{|PD|}$ (Fig. 15). Hence $|PC| = |PD|$. Similarity of triangles APD and BPC implies $\dfrac{|AD|}{|BC|} = \dfrac{|PD|}{|PC|} = 1$, thus $|AD| = |BC|$ as needed.

Solution 2: By assumptions, triangles APD and BPC are similar. Thus $\angle ADB = \angle ACB$ (Fig. 16), showing that quadrilateral $ABCD$ is cyclic. But if a quadrilateral with parallel opposite sides has a circumcircle, the bisectors of these sides coincide as they have the same direction and both pass through the circumcenter of the quadrilateral (Fig. 17). By symmetry w.r.t. this line, the other pair of opposite sides have equal lengths.

F-5. *(Grade 10.)* Each unit square in a 5×5 table is coloured either blue or yellow. Prove that there exists a rectangle with sides parallel to the edges of the table, such that the four unit squares in its corners have the same colour.

Solution 1: Each row contains at least 3 squares with the same colour. Similarly, the dominating colour must be the same in at least 3 rows. W.l.o.g., suppose that the first 3 rows contain at least 3 blue squares each. If the first two rows contain two blue squares in the same columns then the desired rectangle exists. Otherwise, each column contains at least one blue square in these two rows (in Fig. 18, the blue squares are consecutive w.l.o.g.). Thus in one of the first two rows there are two blue squares that are in the same columns with the blue squares in the third row. These form the desired figure.

Solution 2: The first row containes three squares of the same colour, say, blue. If in some of the remaining four rows there are two blue squares aligned to the blue squares in the first row then the desired rectangle exists. Otherwise, each of these four rows contains two yellow squares aligned with the blue squares in the first row. But two yellow squares can be placed into three columns in 3 different ways only. Thus there exist two rows where these two yellow squares are in the same columns. These squares form the desired figure.

F-6. *(Grade 10.)* Jüri draws a circle c with radius 3 and n circles with radius 1 on a paper. Find the minimal n for which he can draw the circles in such a way that it would not be possible to draw inside the circle with radius 3 any new circles with radius 1 having at most one common point with each of the previously drawn circles.

Answer: 3.

Solution: Let O be the center of circle c. Suppose that circles c_1, c_2, c_3 with radius 1 and centers O_1, O_2, O_3, respectively, are placed in such a way that they touch circle c internally and O_1, O_2, O_3 are vertices of an equilateral triangle (coloured dark in Fig. 19). No new circle can be placed to the same line with two existing circles with radius 1 because the three circles would require free area with length 6 which is possible only along the diameter of circle c. The other ways of placement are in the middle or to the other side of the narrower area between two circles (uncoloured in Fig. 19).

The circle with radius 1 and center O touches all three circles, hence the location is fixed. The circle with radius 1 and center O_1' being symmetric to point O w.r.t. line O_2O_3 touches circles c_2 and c_3 and has at least one common point also with circle c, as it is in one line with the circle in the middle and circle c_1. Hence this location is fixed, too. Note that after moving the circle c_2 towards point O the circle will intersect the circle with center O_1' (since $\angle OO_2O_1' < 90°$) and also the circle in the middle. Thus if all circles c_1, c_2, c_3 are moved a bit towards point O, chances to add new circles disappear.

Consequently, if $n \geq 3$ then Jüri can draw the circles in such a way that new circles cannot be added in the required fashion. Show now that in the case $n \leq 2$ a new circle can always be added; it suffices to consider the case $n = 2$. Let c_1 and c_2 be the given circles with radius 1 and centers O_1 and O_2. W.l.o.g., $O_1 \neq O$. Choose AB as the diameter of c which is perpendicular to O_1O (Fig. 20). Consider two circles with radius 1, touching the circle c at points A and B, respectively. Circle c_1 does not preclude drawing either of them as it is located inside the strip with width 2 surrounding the line O_1O (dark in Fig. 20), where neither of the two circles outreach. Circle c_2 can preclude at most one of the two circles since it cannot outreach to both sides of the strip.

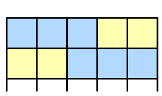

Figure 18

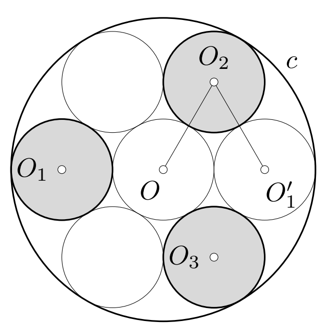

Figure 19

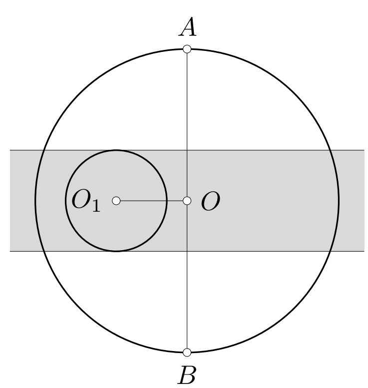

Figure 20

F-7. *(Grade 11.)* Let a finite decimal fraction be given. Juku starts appending digits to this fraction in such a way that each new digit equals the remainder of the sum of all digits existing so far in division by 10. (For instance, if the initial fraction is 27.35 then the digits added to the end are 7, 4, 8 etc.)

Prove that the infinite decimal fraction obtained this way represents a rational number.

Solution: It suffices to show that the infinite decimal fraction is periodic. For that, note that each new digit except for the first digit is congruent to twice the previous digit modulo 10. Indeed, let $a_1, \ldots, a_{k-1}, a_k$ be the existing at some time moment digits where a_k is already added by Juku. Then the next digit a_{k+1} satisfies

$$a_{k+1} \equiv a_1 + \ldots + a_k = (a_1 + \ldots + a_{k-1}) + a_k \equiv a_k + a_k = 2a_k \pmod{10} .$$

Hence each new digit is uniquely determined by the last existing digit. As there are only a finite number of different digits, some digit must be added repeatedly. According to the fact just proven, all following digits are repeated as well.

Remark: This solution can be reformulated without mentioning the fact that each new digit except for the first is congruent to twice the previous one. After noting that Juku must add some digit repeatedly, denote their position numbers in the decimal fraction by m and n. Hence the digits before the mth digit and the digits before the nth digit sum up to congruent numbers modulo 10. Adding the mth and the nth digit, respectively, to the sums maintains the congruence. This means that the next digits are also equal. Thus the digits start repeating periodically.

F-8. *(Grade 11.)* Let $n > 1$ be an integer and $a_1, a_2, \ldots, a_n$ some real numbers, the sum of which is 0 and the sum of the absolute values of which is 1. Prove that

$$|a_1 + 2a_2 + \ldots + na_n| \leq \frac{n-1}{2} .$$

Solution 1. According to the assumptions, for each $k = 1, \ldots, n-1$ it holds that

$$|a_1 + \ldots + a_k| = |a_{k+1} + \ldots + a_n| ,$$

$$|a_1 + \ldots + a_k| + |a_{k+1} + \ldots + a_n| \leq |a_1| + \ldots + |a_k| + |a_{k+1}| + \ldots + |a_n| = 1 .$$

Consequently, $|a_{k+1} + \ldots + a_n| \leq \dfrac{1}{2}$ for each $k = 1, \ldots, n-1$. Now

$$\begin{aligned} & |a_1 + 2a_2 + \ldots + na_n| \\ &\quad = \quad |(a_1 + \ldots + a_n) + (a_2 + \ldots + a_n) + \ldots + (a_{n-1} + a_n) + a_n| \\ &\quad \leq \quad |a_1 + \ldots + a_n| + |a_2 + \ldots + a_n| + \ldots + |a_{n-1} + a_n| + |a_n| \\ &\quad \leq \quad 0 + \tfrac{1}{2} + \tfrac{1}{2} + \ldots + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{n-1}{2} . \end{aligned}$$

Solution 2. Let A^+ and A^- be the sum of positive numbers and the sum of negative numbers, respectively. By the assumptions, $A^+ + A^- = 0$ and $A^+ - A^- = 1$, implying $A^+ = \dfrac{1}{2}$ and $A^- = -\dfrac{1}{2}$.

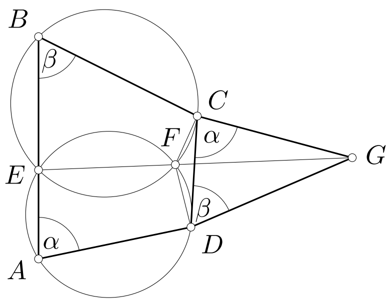

Figure 21

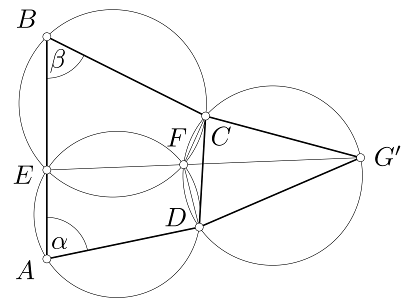

Figure 22

By increasing the coefficients of positive terms and decreasing the coefficients of negative terms, the whole sum can only increase, and by decreasing the coefficients of positive terms and increasing that of negative terms, the sum can only decrease. Thus

$$\begin{aligned} a_1 + 2a_2 + \ldots + na_n &\leq n \cdot A^+ + 1 \cdot A^- = \tfrac{n-1}{2} , \\ a_1 + 2a_2 + \ldots + na_n &\geq 1 \cdot A^+ + n \cdot A^- = -\tfrac{n-1}{2} . \end{aligned}$$

Consequently, $|a_1 + 2a_2 + \ldots + na_n| \leq \dfrac{n-1}{2}$.

Remark: The bound $\dfrac{n-1}{2}$ can be achieved for every n by taking $a_1 = -\dfrac{1}{2}$, $a_2 = \ldots = a_{n-1} = 0$ and $a_n = \dfrac{1}{2}$.

F-9. *(Grade 11.)* A convex quadrilateral $ABCD$ where $\angle DAB + \angle ABC < 180^\circ$ is given on a plane. Let E be a point different from the vertices of the quadrilateral on the line interval AB such that the circumcircles of triangles AED and BEC intersect inside the quadrilateral $ABCD$ at point F. Point G is defined so that $\angle DCG = \angle DAB$, $\angle CDG = \angle ABC$ and triangle CDG is located outside quadrilateral $ABCD$. Prove that the points E, F, G are collinear.

Solution: Denote $\angle DAB = \alpha$ and $\angle ABC = \beta$ (Fig. 21). From cyclic quadrilaterals $AEFD$ and $BEFC$ one obtains

$$\begin{aligned} &\angle DFE = 180^\circ - \angle DAE = 180^\circ - \alpha , \\ &\angle CFE = 180^\circ - \angle CBE = 180^\circ - \beta , \end{aligned}$$

respectively. Thus $\angle CFD = 360^\circ - (180^\circ - \alpha) - (180^\circ - \beta) = \alpha + \beta$. But $\angle CGD = 180^\circ - (\alpha + \beta)$ by the choice of G. Hence the quadrilateral $CFDG$ is cyclic. Consequently, $\angle DFG = \angle DCG = \alpha = 180^\circ - \angle DFE$, which implies that the points E, F, G are collinear.

Remark 1: The argumentation can also be turned around in the following way: Let G' be defined as the other intersection point of the circumcircle of triangle CFD and line EF (Fig. 22). Then the quadrilateral $CFDG'$ is cyclic, whence

$$\begin{aligned} &\angle DCG' = \angle DFG' = 180^\circ - \angle DFE = \alpha , \\ &\angle CDG' = \angle CFG' = 180^\circ - \angle CFE = \beta . \end{aligned}$$

These equalities imply that $G' = G$. Thus G belongs to line EF.

Remark 2: The claim of the problem holds also if point F does not have to be inside the quadrilateral $ABCD$. Then G may also be located between E and F.

F-10. *(Grade 11.)* A $(2k+1) \times (2k+1)$ table, where k is a positive integer, contains one real number in each entry, where these numbers are pairwise different. After each row, one writes the *median* of the row, i.e., the number occurring in this row such that the row contains the same amount of numbers less than it and greater than it. Let m be the median of the column of medians. Prove that more than a quarter of the numbers initially in the table are less than m.

Solution: Each row contains k numbers less than the median and k numbers greater than the median. Thus $k+1$ numbers in each row do not exceed the median of that row. In rows whose median does not exceed m, these $k+1$ numbers do not exceed m either. There are $k+1$ such rows. Consequently, there are at least $(k+1)^2$ numbers in the table that do not exceed m. Only one of them is equal to m, whence $(k+1)^2 - 1 = k^2 + 2k$ numbers are less than m. As k is positive by assumption, we have $1 < 4k$ and $4k + 1 < 4k + 4k = 8k$. Now $\frac{k^2+2k}{(2k+1)^2} = \frac{k^2+2k}{4k^2+4k+1} > \frac{k^2+2k}{4k^2+8k} = \frac{1}{4}$ and we are done.

F-11. *(Grade 11.)* For which natural numbers $n \geq 3$ is it possible to cut a regular n-gon into smaller pieces with regular polygonal shape? (The pieces may have different number of sides.)

Answer: 3, 4, 6, 12.

Solution: A regular triangle can be partitioned into four regular triangles of equal size (Fig. 23), a regular quadrilateral can be partitioned into four regular quadrilaterals with equal size (Fig. 24) and a regular hexagon can be partitioned into six regular triangles of equal size (Fig. 25). By building alternately equilateral triangles and squares onto the sides of a regular 12-gon, just a regular hexagon remains (Fig. 26), whence also a regular 12-gon can be partitioned in the required way.

Show now that other regular polygons cannot be partitioned into smaller regular polygons. For that, consider an arbitrary polygon that is partitioned into regular polygons. As the size of an internal angle of a regular polygon is less than 180° and not less than 60°, at most two regular polygons can meet at each vertex.

If a vertex of the big n-gon is filled by just one smaller polygon then this piece is an n-gon itself. Beside it, there must be space for at least one regular polygon. No more than two regular polygons can be placed there since the sum of the internal angles of these polygons and the n-gon itself would exceed 180°. Two new pieces can be placed only if all these three pieces are triangular, which gives $n = 3$. It remains to study the case where there is exactly one polygon beside the n-gonal piece. The size of the internal angle of the n-gon being at most 120° implies $n \leq 6$. The case $n = 5$ is impossible as its external angles are of size 72° but no regular polygon has internal angles of size strictly between 60° and 90°.

If each vertex of the big n-gon is the meetpoint of two smaller regular polygons then one of them must be a triangle since other regular polygons have internal angles of size 90° or more. Beside a triangle, there is space for a triangle, a quadrilateral or a pentagon.

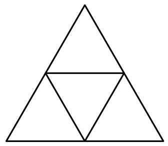

Figure 23

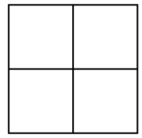

Figure 24

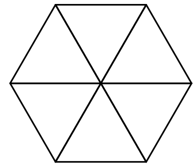

Figure 25

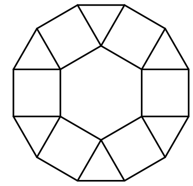

Figure 26

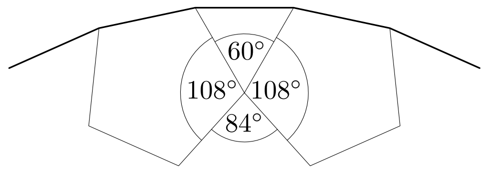

Figure 27

In the first two cases, the size of the internal angles of the n-gon will be $120°$ and $150°$, respectively, covering the cases $n = 6$ and $n = 12$. It remains to show that the third case with a triangle and a pentagon meeting at each vertex is impossible. Indeed, the side length of the pentagon must coincide with the side length of the initial big n-gon, because it is impossible to place a regular polygon beside the pentagon along one side. For the same reason, another pentagon must be built to the second next side along the boundary of the initial polygon. These two pentagons meet at the third vertex of the triangle built to the side between (Fig. 27). But the ulterior angle between the sides of the pentagons at the meeting point has size $360° - 2 \cdot 108° - 60° = 84°$, which cannot be filled with interior angles of regular polygons.

F-12. *(Grade 12.)* Find the smallest natural number n for which there exist integers $a_1, \ldots, a_n$ (that do not have to be different) such that $a_1^4 + \ldots + a_n^4 = 2013$.

Answer: 14.

Solution: Note that the fourth powers of even numbers are divisible by 16 and the fourth powers of odd numbers are congruent to 1 modulo 16. As $2013 \equiv 13 \pmod{16}$, the desired representation must contain at least 13 odd summands.

Suppose that no more summands are needed. As $7^4 = 2401 > 2013$, each summand must be $1^4 = 1$, $3^4 = 81$ or $5^4 = 625$. There can be at most 3 summands 625 since $4 \cdot 625 > 2013$. Therefore the number of summands not divisible by 5 is at least 10. The fourth power of an integer not divisible by 5 is congruent to 1 modulo 5, whereas $2013 \equiv 3 \pmod{5}$. Hence the number of summands not divisible by 5 must be at least 13. This shows that the representation contains only summands 1 and 81, but 13 such numbers sum up to at most $13 \cdot 81$ which is less than 2013. Thus representations with 13 summands are impossible.

On the other hand, 14 fourth powers is enough as $6^4 + 5^4 + 3^4 + 11 \cdot 1^4 = 2013$.

Remark: The fact that 2013 cannot be represented as the sum of 13 odd fourth powers can also be proved without calculations modulo 5. Suppose that

$$a_1^4 + \ldots + a_{13}^4 = 2013\,, \tag{5}$$

where $a_i = 2b_i + 1$ for every $i = 1, \ldots, 13$. We may assume that each b_i is 0, 1 or 2. As

$$(2x+1)^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1 = 16\left(x^4 + 2x^3 + \frac{3}{2}x^2 + \frac{1}{2}x\right) + 1$$

and

$$x^4 + 2x^3 + \frac{3}{2}x^2 + \frac{1}{2}x = x^4 + 2x^3 + x^2 + \frac{1}{2}x^2 + \frac{1}{2}x = x^2(x+1)^2 + \frac{1}{2}x(x+1)\ ,$$

the equality (5) reduces to $f(b_1) + f(b_2) + \ldots + f(b_{13}) = \dfrac{2013 - 1 - 1 - \ldots - 1}{16} = 125$ where $f(x) = x^2(x+1)^2 + \frac{1}{2}x(x+1)$. Thus 125 should be representable as the sum of 13 integers, each of which is $f(0) = 0$, $f(1) = 5$ or $f(2) = 39$. Obviously the number of summands 39 is at most 3 and, as $125 - 13 \cdot 5 = 60$, at least 2. The cases with two and three summands 39 give no solution.

F-13. *(Grade 12.)* Real numbers x_1, x_2, x_3, x_4 in $[0;1]$ are such that the product

$$K = |x_1 - x_2| \cdot |x_1 - x_3| \cdot |x_1 - x_4| \cdot |x_2 - x_3| \cdot |x_2 - x_4| \cdot |x_3 - x_4|$$

is as large as possible. Prove that $\dfrac{1}{27} > K > \dfrac{4}{243}$.

Solution 1: If some two numbers among x_1, x_2, x_3, x_4 are equal then $K = 0$ which is not maximal. Thus assume w.l.o.g. that $x_1 > x_2 > x_3 > x_4$. Applying AM-GM for $x_1 - x_2$, $x_2 - x_3$ and $x_3 - x_4$ gives

$$\sqrt[3]{(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)} \le \frac{(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4)}{3} = \frac{x_1 - x_4}{3} \le \frac{1}{3}\ ,$$

i.e., $|x_1 - x_2| \cdot |x_2 - x_3| \cdot |x_3 - x_4| \le \dfrac{1}{27}$. Among the remaining factors $|x_1 - x_3|$, $|x_1 - x_4|$, $|x_2 - x_4|$, at least one is less than 1. Hence we conclude the left-hand inequality needed.

For the second inequality, note that if $x_1 = 1$, $x_2 = \dfrac{3}{4}$, $x_3 = \dfrac{1}{4}$, $x_4 = 0$ then

$$K = \frac{1}{4} \cdot \frac{3}{4} \cdot 1 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{512} > \frac{4}{243}\ ,$$

since $9 \cdot 243 = 2187 > 2048 = 4 \cdot 512$.

Solution 2: W.l.o.g., assume the inequalities $x_1 > x_2 > x_3 > x_4$. In addition, assume that $x_1 = 1$ and $x_4 = 0$ as otherwise K can be made larger. Substituting $x_2 = y$ and $x_3 = z$ for simplicity, one obtains

$$K = (1-y)(1-z)(y-z)yz = (y-z) \cdot (1-y)z \cdot (1-z)y\ .$$

Consider pairs (y,z) with $y - z$ fixed. The sums $(1-y) + z = 1 - (y-z)$ and $(1-z) + y = 1 + (y-z)$ are then also fixed. The product of two numbers with fixed sum is the largest if the numbers are equal; thus the product $(1-y)z$ is the largest in the case $y+z = 1$ and the product $(1-z)y$ is the largest in the same case $y+z = 1$. Consequently, also K obtains its largest value in the case $y+z=1$. Substituting $1-z$ at place of y, one gets $K = z^2(1-z)^2(1-2z) = (z(1-z))^2(1-2z)$.

Let $f(z) = (z(1-z))^2(1-2z)$, then $f'(z) = 2z(1-z)(1-5z+5z^2)$. The roots of f' within $(0,1)$ are the roots of the quadratic polynomial $5z^2-5z+1$, namely $\frac{5-\sqrt{5}}{10}$ and $\frac{5+\sqrt{5}}{10} = 1 - \frac{5-\sqrt{5}}{10}$. As $f(0) = f\left(\frac{1}{2}\right) = 0$ and $f(z) > 0$ whenever $0 < z < \frac{1}{2}$, the maximum of f is achieved at $z = \frac{5-\sqrt{5}}{10}$. Thus the maximum value of K is $f\left(\frac{5-\sqrt{5}}{10}\right) = \frac{\sqrt{5}}{125}$. This number $\frac{\sqrt{5}}{125}$ satisfies both inequalities of the problem.

F-14. *(Grade 12.)* The midpoints of sides C_2C_3, C_3C_1 and C_1C_2 of a triangle $C_1C_2C_3$ are K_1, K_2 and K_3, respectively. The centers of circles c_1, c_2 and c_3 are C_1, C_2 and C_3, respectively, and the centers of circles k_1, k_2, k_3 are K_1, K_2, K_3, respectively. No two of the given six circles intersect in two points nor are they inside each other. Circles k_1, k_2 and k_3 touch each other externally.

(i) Prove that the sum of the radii of circles c_1, c_2 and c_3 does not exceed one quarter of the perimeter of the triangle $C_1C_2C_3$.

(ii) Prove that if the sum of the radii of circles c_1, c_2 and c_3 equals one quarter of the perimeter of the triangle $C_1C_2C_3$ then the triangle $C_1C_2C_3$ is equilateral.

Solution: Let the radii of the circles c_1, c_2, c_3 be r_1, r_2, r_3, and the radii of the circles k_1, k_2, k_3 be R_1, R_2, R_3, respectively (Fig. 28). By assumptions,

$$\begin{aligned} R_1 + R_2 &= |K_1K_2| = \tfrac{1}{2}|C_1C_2|, \\ R_2 + R_3 &= |K_2K_3| = \tfrac{1}{2}|C_2C_3|, \\ R_3 + R_1 &= |K_3K_1| = \tfrac{1}{2}|C_3C_1|, \end{aligned}$$

which sum up to $2R_1 + 2R_2 + 2R_3 = \frac{1}{2}(|C_1C_2| + |C_2C_3| + |C_3C_1|)$. The assumptions also imply inequalities

$$\begin{aligned} r_1 + R_3 &\le \tfrac{1}{2}|C_1C_2|, & R_3 + r_2 &\le \tfrac{1}{2}|C_1C_2|, \\ r_2 + R_1 &\le \tfrac{1}{2}|C_2C_3|, & R_1 + r_3 &\le \tfrac{1}{2}|C_2C_3|, \\ r_3 + R_2 &\le \tfrac{1}{2}|C_3C_1|, & R_2 + r_1 &\le \tfrac{1}{2}|C_3C_1|, \end{aligned}$$

which sum up to $2r_1 + 2r_2 + 2r_3 + 2R_1 + 2R_2 + 2R_3 \le |C_1C_2| + |C_2C_3| + |C_3C_1|$.

(i) Altogether, we obtain the inequality $2r_1 + 2r_2 + 2r_3 \le \frac{1}{2}(|C_1C_2| + |C_2C_3| + |C_3C_1|)$, which implies $r_1 + r_2 + r_3 \le \frac{1}{4}(|C_1C_2| + |C_2C_3| + |C_3C_1|)$ as desired.

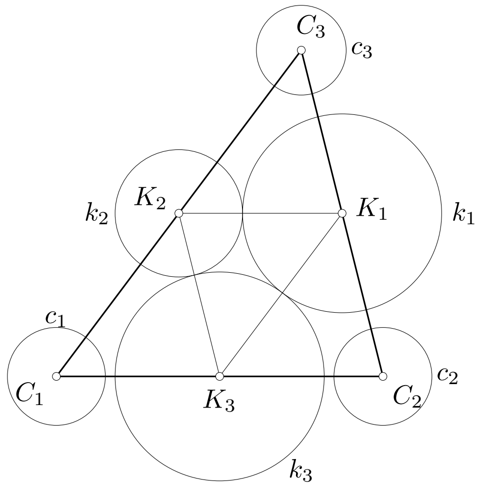

Figure 28

2		**4**
6		**8**

2		**4**
8		**6**

2		**6**
8		**4**

Figure 29

2	9	**4**
7	5	3
6	1	**8**

Figure 30

(ii) Suppose that $r_1 + r_2 + r_3 = \frac{1}{4}(|C_1C_2| + |C_2C_3| + |C_3C_1|)$. Then all inequalities above must hold as equalities. The equalities $r_1 + R_3 = \frac{1}{2}|C_1C_2| = r_2 + R_3$ imply $r_1 = r_2$, analogously $r_1 = r_3$. Denoting $r = r_1 = r_2 = r_3$, we get

$$\begin{aligned} r + R_3 &= \tfrac{1}{2}|C_1C_2| = R_1 + R_2\,, \\ r + R_2 &= \tfrac{1}{2}|C_1C_3| = R_1 + R_3\,, \end{aligned}$$

where summing side-by-side gives $2r + R_2 + R_3 = 2R_1 + R_2 + R_3$, i.e., $R_1 = r$. Analogously, $R_2 = R_3 = r$. Thus all sides of the triangle $C_1C_2C_3$ have length $4r$.

F-15. *(Grade 12.)* Define *magic square* as a 3×3 table where each cell contains one number from 1 to 9 so that all these numbers are used and all row sums and column sums are equal. Prove that any two magic squares can be obtained from each other via the following transformations: interchanging two rows, interchanging two columns, rotating the square, reflecting the square w.r.t. its diagonal.

Solution: As all the transformations are invertible, it suffices to show that every magic square can be turned to one particular magic square by these transformations.

The sum of all numbers in a magic square is 45, whence the numbers in each row and each column must sum up to 15. As this is odd, exactly 0 or 2 of the three summands must be even. There are 4 even numbers in use, hence 2 even numbers must be in some two rows and 0 even number in the remaining one. The same holds for columns.

Hence the even numbers 2, 4, 6, 8 occur in the corners of some rectangle with sides parallel to the edges of the table. By interchanging rows or columns one can move the even numbers to the corners of the whole table. There are 3 possibilities to locate these four numbers into the corners, that can not be obtained from each other by rotations and reflections of the table (Fig. 29). The last two of them cannot occur in the magic square because the missing numbers in the first and third column would coincide. Hence only the first possibility remains. Its completion to a magic square is unique (Fig. 30).

IMO Team Selection Contest

First day

S-1. Find all prime numbers p for which one can find a positive integer m and non-negative integers $a_0, a_1, \ldots, a_m$ less than p such that

$$\begin{cases} a_0 + a_1 p + \ldots + a_{m-1} p^{m-1} + a_m p^m = 2013, \\ a_0 + a_1 + \ldots + a_{m-1} + a_m = 11. \end{cases}$$

Answer: 2003.

Solution: Subtracting the second equation from the first one gives

$$a_1(p-1) + \ldots + a_m\,(p^m - 1) = 2002.$$

As the l.h.s. of the obtained equality is divisible by $p-1$, $2002 = 2 \cdot 7 \cdot 11 \cdot 13$ must also be divisible by $p-1$. Thus $p-1$ equals one of 1, 2, 7, 11, 13, 14, 22, 26, 77, 91, 143, 154, 182, 286, 1001 and 2002. Since p is prime, only 2, 3, 23 and 2003 remain. The first equation of the given system is the p-ary representation of 2013, whence the coefficients a_i are uniquely determined by p.

Now we study all cases.

1. If $p = 2$ then $m = 10$ as $2^{10} < 2013 < 2^{11}$. The second equation implies that all a_is must be ones, but $1 + 2 + 2^2 + \ldots + 2^{10} = 2^{11} - 1 = 2047$. Hence there is no solution in this case.
2. Let $p = 3$. As $2013 = 2 \cdot 3 + 3^2 + 2 \cdot 3^3 + 2 \cdot 3^5 + 2 \cdot 3^6$ whereas $2 + 1 + 2 + 2 + 2 = 9 \neq 11$, this case gives no solution either.
3. Let $p = 23$. As $2013 = 12 + 18 \cdot 23 + 3 \cdot 23^2$ while $12 + 18 + 3 > 11$, this case gives no solution either.
4. For $p = 2003$, we get $2013 = 10 + 2003$ and $10 + 1 = 11$, so the conditions are satisfied.

Consequently, 2003 is the only prime number with the desired property.

S-2. For which positive integers $n \geq 3$ is it possible to mark n points of a plane in such a way that, starting from one marked point and moving on each step to the marked point which is the second closest to the current point, one can walk through all the marked points and return to the initial one? For each point, the second closest marked point must be uniquely determined.

Answer: for all $n \geq 4$.

Solution: To find a construction for any $n \geq 4$, choose $\varepsilon < \dfrac{2\pi}{n^3}$. Place the points A_1, A_2, $\ldots$, A_{n-1} on a circle in such a way that the angle between the radii drawn to the points

A_i and A_{i+1} for $i = 1, \ldots, n-2$ is equal to $\alpha_i = \dfrac{2\pi}{n-2} - (n-2-i)\varepsilon$ (see Fig. 31 for $n = 6$). The angle between the radii drawn to the points A_{n-1} and A_1 is then equal to $\alpha_{n-1} = \dfrac{(n-2)(n-3)}{2}\varepsilon$. Place the point A_n outside of the circle on the extension of the radius containing A_{n-1} at the same distance d from A_{n-1} as the distance between points A_1 and A_2. It is straightforward to verify that the points $A_1, \ldots, A_n$ satisfy the condition of the problem.

On the other hand, suppose that there exists a construction for $n = 3$. Let the cyclic walk be $A_1 \to A_2 \to A_3 \to A_1$. Then $d(A_1, A_2) > d(A_1, A_3)$, $d(A_2, A_3) > d(A_2, A_1)$ and $d(A_3, A_1) > d(A_3, A_2)$, where $d(X, Y)$ denotes the distance between X and Y. But these three inequalities cannot hold simultaneously.

S-3. Let $x_1, \ldots, x_n$ be non-negative real numbers, not all of which are zeros.

(i) Prove that

$$1 \leq \frac{\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots + \frac{x_n}{n}\right) \cdot (x_1 + 2x_2 + 3x_3 + \ldots + nx_n)}{(x_1 + x_2 + x_3 + \ldots + x_n)^2} \leq \frac{(n+1)^2}{4n}.$$

(ii) Show that, for each $n \geq 1$, both inequalities can hold as equalities.

Solution: Applying AM-GM gives

$$\begin{aligned}
\left(\sum_{k=1}^{n} \frac{x_k}{k}\right)\left(\sum_{k=1}^{n} kx_k\right) &= \frac{1}{n} \cdot \left(\sum_{k=1}^{n} \frac{nx_k}{k}\right)\left(\sum_{k=1}^{n} kx_k\right) \leq \\
&\leq \frac{1}{n} \cdot \frac{1}{4}\left(\sum_{k=1}^{n} \frac{nx_k}{k} + \sum_{k=1}^{n} kx_k\right)^2 = \\
&= \frac{1}{4n}\left(\sum_{k=1}^{n} x_k\left(\frac{n}{k} + k\right)\right)^2 \leq \frac{(n+1)^2}{4n}\left(\sum_{k=1}^{n} x_k\right)^2.
\end{aligned}$$

(The last inequality is proved by $\frac{n}{k} + k \leq n + 1$, as it is equivalent to $(n-k)(k-1) \geq 0$.) This gives us the necessary upper bound; this bound is achieved for instance if $x_1 = x_n = 1$ and $x_2 = \ldots = x_{n-1} = 0$.

For the lower bound, estimate the numerator by Cauchy-Schwarz inequality:

$$\left(\sum_{k=1}^{n} \frac{x_k}{k}\right)\left(\sum_{k=1}^{n} kx_k\right) \geq \left(\sum_{k=1}^{n} \sqrt{\frac{x_k}{k}} \cdot \sqrt{kx_k}\right)^2 = \left(\sum_{k=1}^{n} x_k\right)^2;$$

the equality holds here if exactly one of x_is is non-zero.

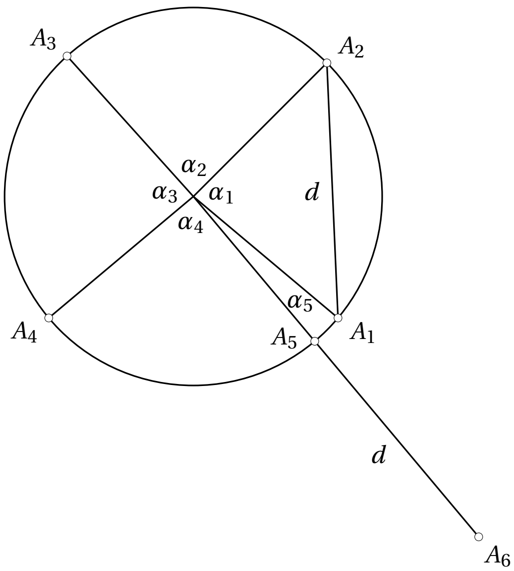

Figure 31

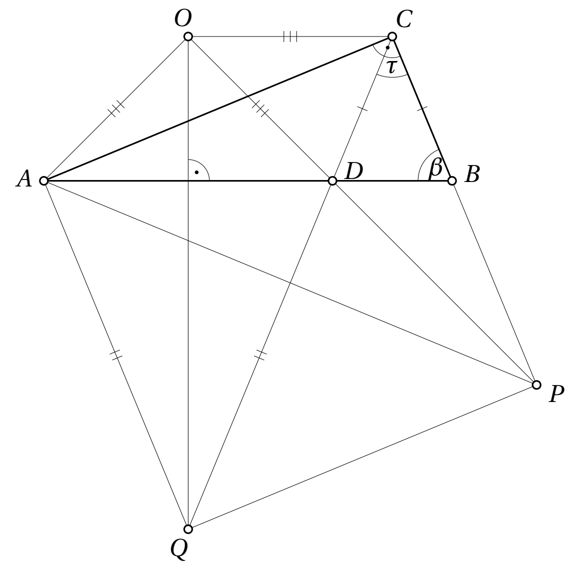

Figure 32

Second day

S-4. Let D be the point different from B on the hypotenuse AB of a right triangle ABC such that $|CB| = |CD|$. Let O be the circumcenter of triangle ACD. Rays OD and CB intersect at point P, and the line through point O perpendicular to side AB and ray CD intersect at point Q. Points A, C, P, Q are concyclic. Does this imply that $ACPQ$ is a square?

Solution: As OQ is the perpendicular bisector of AD, one has $\angle QAD = \angle ADQ = \angle BDC = \angle CBD$ (Fig. 32). Therefore $AQ \parallel BC$, whence $\angle QAC = 180^\circ - \angle ACB = 90^\circ$. From the cyclic quadrilateral $APCQ$ one also gets $\angle CPQ = \angle PQA = 90^\circ$, i.e., $ACPQ$ is a rectangle.

As $\angle DOC = 2\angle DAC$, one obtains

$$\begin{aligned} \angle DOC &= 2\angle BAC = 2(90^\circ - \angle CBA) = 180^\circ - 2\angle CBA = \\ &= 180^\circ - \angle CBD - \angle BDC = \angle DCB\,, \end{aligned}$$

which implies that isosceles triangles BDC and DCO are similar. Thus $\angle BDC = \angle DCO$, i.e., $OC \parallel AB$, whence $\angle QOC = 90^\circ = \angle QAC$. So O lies on the circle determined by A, C, P, Q. Therefore

$$\angle ACQ = \angle AOQ = \frac{1}{2}\angle AOD = \frac{1}{2}\angle AOP = \frac{1}{2}\angle ACP.$$

Consequently, the diagonal of the rectangle $ACPQ$ bisects the angle of the rectangle, whence $ACPQ$ is a square.

Remark: It turns out from the solution that the conditions of the problem determine the shape of the triangle ABC, namely $\angle ABC = \dfrac{3}{8}\pi$.

S-5. Call a tuple $(b_m, b_{m+1}, \dots, b_n)$ of integers *perfect* if both following conditions are fulfilled:

1. There exists an integer $a > 1$ such that $b_k = a^k + 1$ for all $k = m, m+1, \dots, n$;

2. For all $k = m, m+1, \dots, n$, there exists a prime number q and a non-negative integer t such that $b_k = q^t$.

Prove that if $n - m$ is large enough then there is no perfect tuples, and find all perfect tuples with the maximal number of components.

Answer: $\left(2^0 + 1, 2^1 + 1, 2^2 + 1, 2^3 + 1, 2^4 + 1\right)$.

Solution: Clearly $(2^0 + 1, 2^1 + 1, 2^2 + 1, 2^3 + 1, 2^4 + 1)$ is a perfect tuple with length 5. Show in the rest that there are no other perfect tuples with length 5 or larger.

For that, let $(a^m + 1, a^{m+1} + 1, \dots, a^n + 1)$ be an arbitrary perfect tuple with length at least 5. There must exist at least two odd exponents among $m, m+1, \dots, n$; let k and $k+2$ be the two largest odd exponents. As $a^k + 1$ and $a^{k+2} + 1$ are prime powers while having a common divisor $a + 1$, these two integers must be powers of the same prime q. Thus the larger of them, $a^{k+2} + 1$, is divisible by the smaller one, $a^k + 1$, which shows that $a^k + 1$ divides also the difference $a^2 \cdot (a^k + 1) - (a^{k+2} + 1) = a^2 - 1$. Hence $a^k + 1 \le a^2 - 1$, implying $k < 2$. So $k = 1$ as k is odd. By choice of k, the only odd exponents in our perfect tuple are 1 and 3 and the tuple is of the form $(a^0 + 1, a^1 + 1, a^2 + 1, a^3 + 1, a^4 + 1)$.

As $a + 1$ and $a^3 + 1$ are powers of the same prime number q, also the ratio $\dfrac{a^3 + 1}{a + 1} = a^2 - a + 1$ is a power of q. Note that $a^2 - a + 1 \ge 2a - a + 1 = a + 1$ by $a \ge 2$, hence $a^2 - a + 1$ is divisible by $a + 1$. Thus the difference $(a^2 - a + 1) - (a + 1)(a - 2) = 3$ is divisible by $a + 1$. This filters out the only possibility $a = 2$.

S-6. A class consists of 7 boys and 13 girls. During the first three months of the school year, each boy has communicated with each girl at least once. Prove that there exist two boys and two girls such that both boys communicated with both girls first time in the same month.

Solution 1: Call the first communication between a boy and a girl their *acquaintance*. During the 3 months, there are $7 \cdot 13 = 91$ acquaintances in total. Thus there exists a month when there was at least 31 aquaintances. Let the boys be denoted by p_1 through p_7 and let T_i, $i = 1, \dots, 7$, be the set of girls to whom p_i acquainted in this month. We have to show that there exist distinct i and j such that $T_i \cap T_j$ contains at least 2 girls. W.l.o.g., assume the inequalities $|T_1| \ge |T_2| \ge \dots \ge |T_7|$. Consider two cases.

1. *The case* $|T_1| + |T_2| + |T_3| + |T_4| \ge 20$. Suppose that all intersections $T_1 \cap T_2, T_1 \cap T_3, \dots, T_3 \cap T_4$ contain at most one girl. Let $k \le 6$ be the number of non-empty intersections. Then the first four boys acquainted with

$$|T_1 \cup T_2 \cup T_3 \cup T_4| \ge |T_1| + |T_2| + |T_3| + |T_4| - k \ge 20 - 6 = 14$$

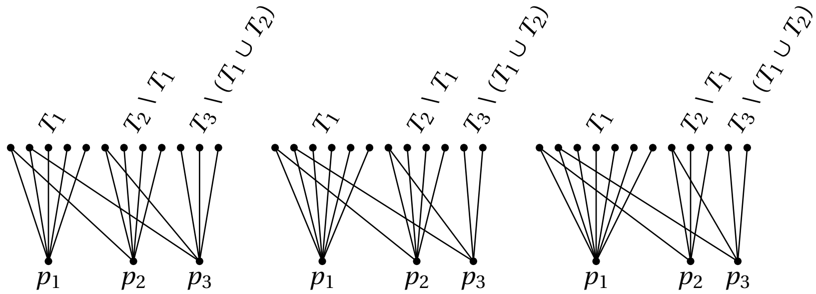

Figure 33

girls in the month under consideration. (For proving the first inequality, note that girls that belong to one or two subsets count once in the r.h.s., girls belonging to three subsets not count and girls in all four subsets count -2 times.)

This is a contradiction since there are only 13 girls.

2. *The case* $|T_1| + |T_2| + |T_3| + |T_4| \leq 19$. As $|T_5| + |T_6| + |T_7| \geq 12$, we have $|T_5| \geq 4$. Now $|T_4| \leq 4$ implies $|T_4| = |T_5| = 4$ and hence also $|T_6| = |T_7| = 4$. Now $|T_1| + |T_2| + |T_3| = 15$ in order to get 31 in total.

   Suppose that all intersections $T_i \cap T_j$, $i, j = 1, \ldots, 7$, contain at most one girl. If boys p_1, p_2 and p_3 altogether acquainted with all girls in this month then at least one intersection $T_i \cap T_4$, $i = 1, 2, 3$, contains at least two girls. Otherwise, $|T_1| + |T_2 \setminus T_1| + |T_3 \setminus (T_1 \cup T_2)| = 15 - 1 - 2 = 12$ (Fig. 33 shows all possibilities), since only then there is a girl, say t_{13}, with whom none of p_1, p_2, p_3 acquainted. Clearly all boys p_4, p_5, p_6, p_7 must have acquainted with her, and each of them also acquainted with one girl from sets T_1, $T_2 \setminus T_1$ and $T_3 \setminus (T_1 \cup T_2)$. As the last set contains at most three elements, two of the four boys acquainted with the same girl from $T_3 \setminus (T_1 \cup T_2)$.

Solution 2: Let A be the set of all combinations of two boys, $|A| = \binom{7}{2} = 21$. Say that girl t *determines* an element $\{p_1, p_2\}$ of A if t acquainted with p_1 and p_2 in the same month. If in the ith month a girl t acquainted with exactly n_i boys, where $i = 1, 2, 3$, then t determines $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2}$ elements of A. Applying Jensen's inequality for $f(x) = \dfrac{x(x-1)}{2}$ gives $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} \geq 3 \cdot f\left(\dfrac{n_1 + n_2 + n_3}{3}\right) = 3 \cdot f\left(\dfrac{7}{3}\right) = 4\dfrac{2}{3}$. As $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2}$ is an integer, $\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} \geq 5$. Hence all girls determine at least $13 \cdot 5 = 65$ elements of A in total.

As $65 \geq 3 \cdot 21 + 1$, an element $\{p_1^*, p_2^*\}$ of A is determined by at least 4 girls by the pigeonhole principle. Consequently there exists a month in which this couple of boys is determined by at least two girls.

Solution 3: Suppose that there is no required pairs of boys and girls. Like in Solution 1, consider a month with 31 or more acquaintances. Let g_i be the number of girls who acquainted with exactly i boys in this month, $i = 0, 1, \ldots, 7$. We get a system of inequalities

$$\begin{cases} g_2 + 3g_3 + 6g_4 + 10g_5 + 15g_6 + 21g_7 & \leq & 21, \\ g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_7 & \leq & 13, \\ g_1 + 2g_2 + 3g_3 + 4g_4 + 5g_5 + 6g_6 + 7g_7 & \geq & 31. \end{cases} \tag{6}$$

If at least one girl aquainted with 5 or more boys then $g_5 + g_6 + g_7 \geq 1$. The system (6) then reduces to

$$\begin{cases} g_2 + 3g_3 + 6g_4 + 10(g_5 + g_6 + g_7 - 1) & \leq & 11, \\ g_1 + g_2 + g_3 + g_4 + (g_5 + g_6 + g_7 - 1) & \leq & 12, \\ g_1 + 2g_2 + 3g_3 + 4g_4 + 7(g_5 + g_6 + g_7 - 1) & \geq & 24. \end{cases}$$

Summing the first two inequalities gives

$$g_1 + 2g_2 + 4g_3 + 7g_4 + 11(g_5 + g_6 + g_7 - 1) \leq 23,$$

which contradicts the third inequality.

Thus $g_5 = g_6 = g_7 = 0$ and the system of inequalities reduces to

$$\begin{cases} g_2 + 3g_3 + 6g_4 & \leq & 21, \\ g_1 + g_2 + g_3 + g_4 & \leq & 13, \\ g_1 + 2g_2 + 3g_3 + 4g_4 & \geq & 31. \end{cases} \tag{7}$$

Suppose that $g_4 \geq 1$. As in the previous case, (7) implies

$$\begin{cases} g_2 + 3g_3 + 6(g_4 - 1) & \leq & 15, \\ g_1 + g_2 + g_3 + (g_4 - 1) & \leq & 12, \\ g_1 + 2g_2 + 3g_3 + 4(g_4 - 1) & \geq & 27. \end{cases}$$

The first two inequalities sum up to $g_1 + 2g_2 + 4g_3 + 7(g_4 - 1) \leq 27$. In the light of the third inequality, this is possible only if $g_3 = 0$ and $g_4 = 1$. Then the second and third inequalities give $g_1 + g_2 \leq 12$ and $g_1 + 2g_2 \geq 27$, which contradict each other since $g_1 + 2g_2 \leq 2(g_1 + g_2)$.

Hence also $g_4 = 0$ and our system of inequalities reduces to

$$\begin{cases} g_2 + 3g_3 & \leq & 21, \\ g_1 + g_2 + g_3 & \leq & 13, \\ g_1 + 2g_2 + 3g_3 & \geq & 31. \end{cases}$$

Subtracting the first inequality from the third one, we obtain $g_1 + g_2 \geq 10$. Subtracting twice the second inequality from the third one, we get $g_3 - g_1 \geq 5$. The two inequalities obtained sum up to $g_2 + g_3 \geq 15$, contradicting the second inequality.

Remark 1: After decreasing either the number of boys or the number of girls, the claim of the problem would not hold anymore.

Remark 2: This problem is a variant of the problem F-5 from the Final Round for the 10th grade. In terms of that problem, here we take 7×13 table instead of 5×5 and use three colours instead of two.

Problems Listed by Topic

Number theory: O-1, O-5, O-10, O-11, O-15, F-3, F-7, F-12, S-1, S-5

Algebra: O-2, O-6, O-9, O-12, O-16, O-19, F-8, F-13, S-3

Geometry: O-3, O-7, O-13, O-17, F-1, F-4, F-6, F-9, F-14, S-4

Combinatorics: O-4, O-8, O-14, O-18, F-2, F-5, F-10, F-11, F-15, S-2, S-6