



Estonian Math Competitions 2008 / 2009

The Gifted and Talented Development Centre

Tartu 2009

WE THANK:



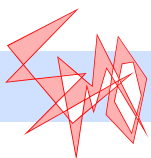
Estonian Ministry of Education and Research

University of Tartu

Problem authors: Juhan Aru, Maksim Ivanov, Oleg Koshik, Toomas Krips,
Kaie Kubjas, Härmel Nestra, Uve Nummert, Oleksandr Rybak (Ukraine),
Laur Tooming, Jan Willemson, Indrek Zolk

Translators: Emilia Käsper, Härmel Nestra, Ago-Erik Riet

Editor: Indrek Zolk



Estonian Mathematical Olympiad

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds – at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organise olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade are allowed to participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

According to the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May. This contest lasts two days; each day, the contestants have 4.5 hours to solve 3 problems, similarly to the IMO. All participants are given the same problems. Some problems in our selection contest are at the level of difficulty of the IMO but somewhat easier problems are usually also included.

The problems of previous competitions can be downloaded from
<http://www.math.olympiadid.ut.ee/eng>.

Besides the above-mentioned contests and the quiz “Kangaroo” some other regional competitions and matches between schools are held as well.

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This booklet contains problems that occurred in the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem to be interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

OC-1. The feet of the altitudes drawn from vertices A and B of an acute triangle ABC are K and L , respectively. Prove that if $|BK| = |KL|$ then the triangle ABC is isosceles. (*Juniors.*)

Solution 1. From $|BK| = |KL|$ we have that BKL is an isosceles triangle and $\angle KBL = \angle KLB$ (Fig. 1). Now $\angle KLC = \frac{\pi}{2} - \angle KLB$, $\angle KCL = \angle BCL = \frac{\pi}{2} - \angle KBL$. Thus $\angle KLC = \angle KCL$. So $|KC| = |KL| = |BK|$. As the altitude AK is now also a median, the triangle ABC is isosceles and $|AB| = |AC|$.

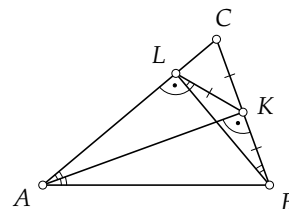


Figure 1

Solution 2. Similarly to Solution 1 $\angle KBL = \angle KLB$. As $\angle AKB = \angle ALB = \frac{\pi}{2}$ (Fig. 1), the points A, B, K and L are concyclic. Now $\angle KAB = \angle KLB = \angle KBL = \angle KAL$ whence the altitude AK is also the bisector of angle CAB . Thus ABC is isosceles.

Remark. The claim can be proven for any triangle.

OC-2. A computer program adds numbers $\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}$, etc., and represents every intermediate sum as a fraction in lowest terms. Prove that for every positive integer k there is a fraction among the results whose denominator is divisible by at least k different primes. (*Seniors.*)

Solution 1. For every positive integer n , denote $a_n = \frac{n!}{1!} + \frac{n!}{2!} + \dots + \frac{n!}{n!}$, then the n -th intermediate sum is $\frac{a_n}{n!}$.

Let k be a fixed positive integer. Let t be such that there are at least $2k - 1$ primes less than or equal to t ; then $t!$ is divisible by at least $2k - 1$ primes. If there are at least k different prime divisors of the denominator of the fraction we get by writing $\frac{a_t}{t!}$ in lowest terms, we have what we were looking for. Otherwise, a_t is divisible by all but at most $k - 1$ prime divisors of $t!$ – there are at least k of those. Note that $a_{t+1} = (t + 1) \cdot a_t + 1$. Thus a_t and a_{t+1} are coprime. So a_{t+1} is not divisible by the prime divisors of $t!$ we considered. Thus, writing $\frac{a_{t+1}}{(t + 1)!}$ in lowest terms, the denominator will have at least k prime divisors, so in this case we also have the fraction we were looking for.

Solution 2. Define a_n as in Solution 1; note that, for every $n > 1$, $a_n = n \cdot a_{n-1} + 1$. Let n be a number divisible by k different primes. Then the n -th intermediate sum is of the form $\frac{n \cdot a_{n-1} + 1}{n!}$. The numerator is obviously coprime with n whence we cannot reduce the fraction by dividing numerator and denominator by any prime divisor of n – there are k of those by construction. The number $n!$ in the denominator is divisible by all of those primes, thus the denominator is divisible by at least k primes even after writing the fraction in lowest terms.

OC-3. Three circles in a plane have the sides of a triangle as their diameters. Prove that there is a point that is in the interior of all three circles. (*Seniors.*)

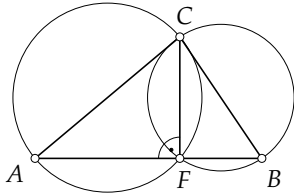


Figure 2

Solution 1. Let the triangle ABC be given. Assume w.l.o.g. that the largest angle is at C (Fig. 2). As the angles at A and B are acute, the foot of the altitude drawn from C – let this be F – lies between A and B . Since CFA is a right angle, the circle with diameter CA goes through point F , thus the altitude CF is a chord to the circle. Analogously, CF is a chord to the circle with diameter CB . Hence all the points on the altitude CF except C and F lie inside both of the circles. Since point F is in the interior of the triangle with diameter AB , there are interior points of the segment FC that are in the interior of that circle.

Solution 2. Given the triangle ABC , let the angles at A, B, C have respectively sizes α, β, γ (Fig. 3). Let I be the intersection of the angle bisectors of ABC . Since $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\alpha + \beta}{2} < \frac{\alpha + \beta + \gamma}{2} = \frac{\pi}{2}$, we have $\angle AIB = \pi - \left(\frac{\alpha}{2} + \frac{\beta}{2}\right) > \pi - \frac{\pi}{2} = \frac{\pi}{2}$, i.e. AIB is an obtuse-angled triangle with the obtuse angle at I . This shows that I is inside the circle with diameter AB . Analogously we show that I is inside both of the other circles.

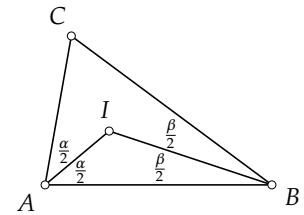


Figure 3

OC-4. There are three ants at vertex A_1 of the regular n -gon $A_1A_2 \dots A_n$ initially. Each minute some two of them move simultaneously to a neighbouring vertex in different directions (one clockwise and the other counter-clockwise), while the third ant stands still. For which n can it happen that after some time all the ants meet at a vertex different from A_1 ? (*Seniors.*)

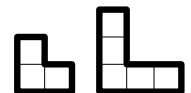
Answer: if n is divisible by 3.

Solution. Note that the sum of numbers of the vertices on which the ants sit is an invariant modulo n . As initially this sum is 3, it will always be 3 modulo n .

We now show that if 3 does not divide n then it is not possible for the ants to meet at a different vertex. Suppose that the ants are all at k after a move. Then $3k$ is congruent to 3 modulo n , or $3k - 3 = 3(k - 1)$ is divisible by n . This is possible only if $k = 1$.

Finally show that if n is divisible by 3 then it is possible to meet at another vertex. Let the first and second ants move $\frac{n}{3}$ times, ending up in $A_{\frac{n}{3}+1}$, resp. $A_{\frac{2n}{3}+1}$. Then the third and second ants move $\frac{n}{3}$ times, taking the third ant to $A_{\frac{n}{3}+1}$ and the second ant to $A_{\frac{n}{3}+1}$. After that all the ants sit at $A_{\frac{n}{3}+1}$.

OC-5. A unit square is removed from the corner of the $n \times n$ grid where $n \geq 2$. Prove that the remainder can be covered by copies of the figures consisting of 3 or 5 unit squares depicted in the drawing. Every square must be covered once and the figures must not go over the bounds of the grid. (*Seniors.*)



Solution. Assume w.l.o.g. that the unit square removed is the one in the bottom right

corner. Further let us write “ $n \times n$ grid” for the grid with the bottom right corner square removed.

The cases $n = 2, 3, 4$ can be done by trial (Figures 4, 5 and 6).

We show how to extend the construction for n to a construction for $n + 3$; by repeating this procedure, we get a solution for any positive integer n .

Let n be odd; consider an $(n + 3) \times (n + 3)$ grid. We cover the $n \times n$ grid in its bottom right corner, then cover the $(n - 1) \times 3$ band on the left and the $3 \times (n - 1)$ band above by 2×3 rectangles formed by two 3-square figures (Fig. 7). We are left with a 4×4 grid in the top left corner which we can already cover.

Let n be even; consider an $(n + 3) \times (n + 3)$ grid. Cover the $n \times n$ grid in its lower right corner, then cover the $(n - 2) \times 3$ band on the left and the $3 \times (n - 2)$ band above by 2×3 rectangles. The remaining part in the top left corner can be covered like shown in Fig. 8.



Figure 4



Figure 5

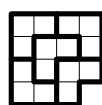


Figure 6



Figure 7

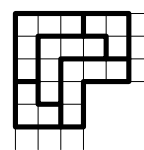


Figure 8

Remark. It is also possible to extend the construction for n to a construction for $n + 2$, dividing into cases according to the remainder of n modulo 3.

OC-6. Find all positive integers n for which there are exactly $2n$ pairs of integers (a, b) where $1 \leq a < b \leq n$ and b is divisible by a . (*Seniors.*)

Answer: 15.

Solution. Denote the number of pairs corresponding to an integer n by $g(n)$. Obviously $g(1) = 0$. Let $n > 1$. The pairs whose second component is at most $n - 1$ have been counted for $n - 1$. We have to add pairs with the second component n and the first component a proper divisor of n . Denoting the number of proper divisors of n by $d(n)$, we have

$$g(n) = g(n - 1) + d(n). \tag{1}$$

We tabulate according to (1).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$d(n)$	0	1	1	2	1	3	1	3	2	3	1	5	1	3	3	4	1
$g(n)$	0	1	2	4	5	8	9	12	14	17	18	23	24	27	30	34	35

We see that only $n = 15$ satisfies the conditions among the numbers considered.

Note that two consecutive numbers greater than 4 have at least 4 proper divisors in total. Indeed, one of them is an even number greater than 4 and has at least 3 proper divisors (1, 2 and half of itself), the other one has at least one proper divisor (number 1). So for every $n \geq 4$ we have $g(n + 2) \geq g(n) + 4$.

It follows that if for some n we have $g(n) > 2n$ then $g(n + 2) \geq g(n) + 4 > 2n + 4 = 2(n + 2)$. Since the inequality holds for $n = 16$ and $n = 17$, we obtain by induction that $g(n) > 2n$ for any $n \geq 16$. Thus there are no more numbers satisfying the condition in the problem.

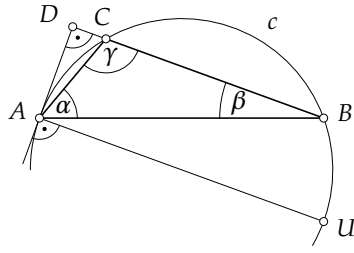


Figure 9

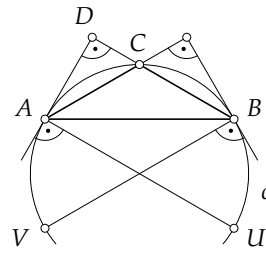


Figure 10

OC-7.

- An altitude of a triangle is also a tangent to its circumcircle. Prove that some angle of the triangle is larger than 90° but smaller than 135° .
- Some two altitudes of the triangle are both tangents to its circumcircle. Find the angles of the triangle. (*Seniors.*)

Answer: b) $120^\circ, 30^\circ, 30^\circ$.

Solution 1. Let our triangle be ABC and the sizes of angles at A, B, C be α, β, γ , respectively (Fig. 9). Assume the altitude of ABC drawn from A is a tangent to the circumcircle; the point of tangency is A since it lies on the circle. Let D be the foot of the altitude. W.l.o.g. assume C is between B and D .

By the tangent-secant theorem, $\angle DAC = \angle DBA = \beta$. The angle ACB is an exterior angle of the triangle ACD , hence $\gamma = \angle ACB = \angle ADC + \angle DAC = 90^\circ + \beta$. Thus $\gamma > 90^\circ$ and also $\beta = \gamma - 90^\circ$.

a) In the triangle ABC we have

$$180^\circ = \alpha + \beta + \gamma = \alpha + 2\gamma - 90^\circ,$$

whence $2\gamma + \alpha = 270^\circ$. Thus $2\gamma < 270^\circ$ which gives $\gamma < 135^\circ$.

b) Obviously the altitude drawn from the vertex of the obtuse angle goes inside the circumcircle and thus cannot be a tangent. So we have to let the altitude drawn from B be a tangent to the circumcircle (Fig. 10). Then in addition to the identity $\beta = \gamma - 90^\circ$ we also have $\alpha = \gamma - 90^\circ$. Thus $180^\circ = 3\gamma - 180^\circ$ whence $\gamma = 120^\circ$ and also $\alpha = \beta = 30^\circ$.

Solution 2. We use notations from the previous solution.

a) Since DA is a tangent and DC is a secant to the same circle drawn from the same point we have $|DA| > |DC|$. Thus $\angle ACD > \angle CAD$. Since $\angle ACD + \angle CAD = 180^\circ - \angle ADC = 180^\circ - 90^\circ = 90^\circ$, it follows that $90^\circ > \angle ACD > \frac{1}{2} \cdot 90^\circ = 45^\circ$. As $\gamma = 180^\circ - \angle ACD$, we obtain $90^\circ < \gamma < 135^\circ$.

b) Let O be the centre of the circumcircle. As $AO \perp AD$ and $BC \perp AD$ we have $AO \parallel BC$. Analogously $BO \parallel AC$. Thus $ACBO$ is a parallelogram, but since $|OA| = |OB|$ it is a rhombus. Since the diagonal OC has the same length as the sides, ACO and BCO are equilateral triangles. Thus $\gamma = \angle ACO + \angle BCO = 60^\circ + 60^\circ = 120^\circ$. The diagonal of a rhombus bisects its angle, thus $\alpha = \beta = \frac{1}{2} \cdot 60^\circ = 30^\circ$.

OC-8. Find all functions f from positive real numbers to positive real numbers such that the curve $y = c \cdot f(x)$ is symmetric with respect to the line $y = x$ for every positive real number c . (*Seniors.*)

Answer: $f(x) = \frac{a}{x}$ where a is an arbitrary positive real number.

Solution. The symmetry of the curve $y = g(x)$ with respect to the line $y = x$ is equivalent to the condition that $g(g(x)) = x$ for each positive real number x . Indeed, the point $(x, g(x))$ lies on the curve $y = g(x)$. By symmetry, the point $(g(x), x)$ is also on the curve – but this is equivalent to $g(g(x)) = x$.

Using this for the curve $y = c \cdot f(x)$, we have for every $c > 0$ and $x > 0$ that

$$c \cdot f(c \cdot f(x)) = x. \tag{2}$$

Taking $c = \frac{1}{f(x)}$ in the identity (2) we have $\frac{1}{f(x)} \cdot f(1) = x$, i.e. $f(x) = \frac{f(1)}{x}$ for every positive real x .

It remains to check that every function of form $f(x) = \frac{a}{x}$ where $a > 0$ satisfies the conditions of the problem. Consider the function g where $g(x) = c \cdot f(x) = \frac{ca}{x}$ and $c > 0$. Now

$$g(g(x)) = g\left(\frac{ca}{x}\right) = \frac{ca}{\frac{ca}{x}} = x,$$

which means that the curve $y = c \cdot f(x)$ is symmetric with respect to the line $y = x$.

Remark. There are other possibilities to determine all possible functions. E.g. taking $x = c = 1$ in the identity (2), we have $f(f(1)) = 1$, yielding by the substitution $x = f(1)$ in (2) that for every positive real number c it holds that $c \cdot f(c) = f(1)$.

Selected Problems from the Final Round of National Olympiad

FR-1. In triangle ABC , points F and E are chosen on sides AC and BC , respectively, such that $2|CF| = |FA|$ and $2|CE| = |EB|$. Outside triangle ABC , points K and L are chosen on rays AE and BF , respectively, such that $2|KE| = |EA|$ and $2|LF| = |FB|$. Prove that $ABKL$ is a parallelogram. (*Grade 9.*)

Solution 1. Note that $\triangle ABF \sim \triangle CLF$ and $\triangle ABE \sim \triangle KCE$ with similarity ratio 2 (Fig. 11). Indeed, $\angle BFA = \angle LFC$ and $\angle BEA = \angle CEK$ (opposite angles) and we know that $|AF| = 2|CF|$, $|BE| = 2|CE|$, $|BF| = 2|LF|$ and $|AE| = 2|KE|$. Thus, LC and CK are parallel to AB and $2|LC| = 2|CK| = |AB|$. Consequently, points L, C and K are collinear, $|LK| = |AB|$ and $LK \parallel AB$, implying that $ABKL$ is a parallelogram.

Solution 2. The assumptions of the problem directly yield:

- $\triangle ABC \sim \triangle FEC$ with similarity ratio 3;
- $\triangle AKC \sim \triangle AEF$ with similarity ratio $\frac{3}{2}$;
- $\triangle BCL \sim \triangle BEF$ with similarity ratio $\frac{3}{2}$.

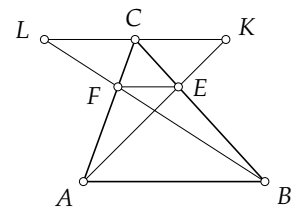


Figure 11

It follows that $3|EF| = |AB|$, $\frac{3}{2}|EF| = |KC|$ and $\frac{3}{2}|EF| = |CL|$, where EF (and thus also KC and CL) are parallel to AB . Consequently, K , C and L are collinear, $KL \parallel AB$, and $|KL| = |KC| + |CK| = 3|EF| = |AB|$, implying that $ABKL$ is a parallelogram.

Solution 3. Let lines BC and AL intersect at N , and let lines AC and BK intersect at M . Since $\frac{BF}{FL} = \frac{AF}{FC} = 2$, we have that BL and AC are medians of ABN and LC is the mid-segment of side AB : $LC \parallel AB$ and $|AB| = 2|LC|$. Similarly, BC and AK are medians of ABM and CK is the mid-segment of side AB : $CK \parallel AB$ and $|AB| = 2|CK|$. Hence, L , C and K are collinear, $LK \parallel AB$ and $|LK| = |AB|$, implying that $ABKL$ is a parallelogram.

FR-2. Call a positive integer m *magic* if the sum of its digits equals the product of its digits.

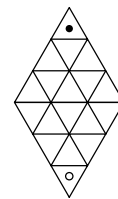
- Prove that for all $n = 1, 2, \dots, 10$, there exists a magic number consisting of precisely n digits.
- Prove that there exist infinitely many magic numbers. (*Grade 9.*)

Solution 1. a) The numbers 1, 22, 123, 1124, 11125, 111126, 1111127, 11111128, 111111129 and 1111111144 are magic.

b) Given any positive integer for which the product of its digits is larger than the sum, we can construct a magic number by appending a suitable number of 1-digits: each appended 1 increases the sum by 1 without changing the product. Now, for any $n > 0$, the product of the digits of $\underbrace{22 \dots 2}_n$ equals $2^n \geq 2n$, the sum of the digits, so for any n , we can construct a magic number with at least n digits.

Solution 2. b) Let $m > 1$ be a magic number. We can always construct a larger magic number m' by appending 2 to m , followed by a suitable number of 1-digits, as appending a 2 increases the sum of the digits by 2 and the product of the digits by at least 2, while appending a 1 increases the sum by 1 without changing the product.

FR-3. Juku and Miku play a game on a rhombus of side length n consisting of two equilateral triangles divided into equilateral triangular tiles with side length 1 ($n = 3$ in the figure). Each player has one token. At the beginning of the game, the tokens lie on the topmost and bottommost tile, respectively. Players alternate moves by sliding their token one step to an adjacent tile (tiles are adjacent if they share a side). A player wins the game by capturing his opponent's token (moving his own token to the same tile where the opponent's token lies); or by reaching his opponent's starting tile. Suppose Juku makes the first move. Does either of the players (who?) have a winning strategy? (*Grade 9.*)



Answer: Juku has a winning strategy.

Solution. We show that Miku cannot capture Juku's token. Colour the tiles with the triangle "pointing upwards" black, and the remaining tiles white, then any two adjacent tiles are of different colour and every move takes the token from a black tile to a white tile or vice versa. As the starting tiles are of different colour, after every two moves, the tokens again lie on tiles of different colour and no Miku's move can result in capturing Juku's token.

Consequently, Juku can safely take the shortest route to his opponent's starting tile. Since the shortest route has equal length for Juku and Miku, Juku is bound to reach his goal first.

FR-4. Find all pairs of positive integers (m, n) such that in an $m \times n$ rectangular grid, the number of unit squares touching at least one side of the rectangle equals the number of remaining unit squares. (Grade 9.)

Answer: $(5, 12), (6, 8), (8, 6), (12, 5)$.

Solution. We can exclude $m = 1$ or $n = 1$, as then all unit squares touch a side.

Assume now $m \geq 2, n \geq 2$. There are $2m + 2n - 4$ unit squares touching a side, so the number of remaining squares is $mn - 2m - 2n + 4$. We require $2m + 2n - 4 = mn - 2m - 2n + 4$, which is equivalent to $(m - 4)(n - 4) = 8$. As $m - 4 \geq -2$ and $n - 4 \geq -2$, we can eliminate $\{-1, -8\}$ and $\{-2, -4\}$ as possible factorings of 8. The remaining possibilities $\{1, 8\}$ and $\{2, 4\}$ yield four solutions for (m, n) .

FR-5. Let the angles of a triangle measure x, y, z in degrees.

- Prove that if $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$ are all rational, then x, y, z are also all rational.
- Prove that if exactly one of $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$ is rational, then x, y, z are all irrational.

(Grade 10.)

Solution. Note that

$$\frac{180}{x} = \frac{x + y + z}{x} = \frac{x}{x} + \frac{y}{x} + \frac{z}{x} = 1 + \frac{y}{x} + \frac{z}{x}. \quad (3)$$

a) Assume $\frac{y}{x} = \frac{1}{\frac{x}{y}}$ and $\frac{z}{x}$ are rational. By (3), $\frac{180}{x}$ is a sum of three rationals and thus itself rational. Hence, x is rational. The proof for y and z is analogous.

b) Assume w.l.o.g. $\frac{x}{y}$ (and thus also $\frac{y}{x}$) is rational and $\frac{y}{z}, \frac{z}{x}$ (and thus also $\frac{z}{y}, \frac{x}{z}$) are irrational. The equation (3) then represents $\frac{180}{x}$ as a sum of one irrational and two rational numbers. Hence, $\frac{180}{x}$ is irrational, and so is x . As $\frac{x}{y}$ is rational, y must also be irrational.

Assume now z is rational, then $x + y = 180 - z$ is also rational. Now $\frac{x + y}{y}$ as a ratio of a rational and an irrational number is irrational, yet $\frac{x + y}{y} = \frac{x}{y} + 1$ as a sum of two rationals must be rational, contradiction.

FR-6. Find all triples of positive integers (x, y, z) satisfying $99x + 100y + 101z = 2009$. (Grade 10.)

Answer: $(1, 9, 10), (2, 7, 11), (3, 5, 12), (4, 3, 13), (5, 1, 14)$.

Solution. We bound the lhs from both sides:

$$\begin{aligned} 2009 &= 99x + 100y + 101z \leq 101(x + y + z), \\ 2009 &= 99x + 100y + 101z \geq 99(x + y + z). \end{aligned}$$

Rearranging, we get $19 < \frac{2009}{101} \leq x + y + z \leq \frac{2009}{99} < 21$, implying $x + y + z = 20$.

Now, rewriting the original equation as $100(x + y + z) + z - x = 2009$ gives $z = x + 9$. Substituting z in $x + y + z = 20$ then gives $y = 11 - 2x$. Since x and y are positive integers, we must have $0 < x \leq 5$. The values 1, 2, 3, 4, 5 of x lead to solutions (1, 9, 10), (2, 7, 11), (3, 5, 12), (4, 3, 13), (5, 1, 14) of the equation, respectively.

FR-7. In an acute triangle ABC , draw a perpendicular y to AB through B , and a perpendicular z to AC through C . Prove that the intersection point of y and z lies on the perpendicular drawn to BC through A iff $|AB| = |AC|$. (Grade 10.)

Solution 1. Let the perpendicular drawn to BC through A be x , and let x intersect BC at E (Fig. 12). Assume x , y and z intersect at D , then from right triangles ABD and ACD , $|AB| = \sqrt{|AD| \cdot |AE|} = |AC|$.

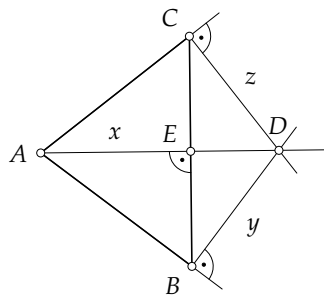


Figure 12

Conversely, assume $|AB| = |AC|$, then triangle ABC is isosceles with vertex angle A . Altitude AE bisects the vertex angle, so segments AB and AC are symmetric wrt x . But then also perpendiculars y and z drawn through B and C are symmetric wrt x , and their intersection points with x coincide.

Solution 2. Let the notations be as above. Since AB is perpendicular to BD and AC to CD , points A, B, C, D lie on the same circle with diameter AD . As chord BC is perpendicular to AD , the intersection point E bisects BC . Hence, AE is both an altitude and a median in triangle ABC , so $|AB| = |AC|$.

Conversely, assume $|AB| = |AC|$, then $\angle ACB = \angle ABC$ and in triangle ABC , the altitude drawn from vertex A coincides with the angle bisector. Let y and z intersect at F . Since $\angle BAF = \angle FCB = \frac{\pi}{2} - \angle ACB = \frac{\pi}{2} - \angle ABC = \angle CBF = \angle CAF$, AF is also an angle bisector. Thus, line AF coincides with x , so x , y and z intersect at F .

FR-8. Mari and Jüri play a game on an $2 \times n$ rectangular grid ($n > 1$) whose sides of length 2 are glued together to form a cylinder. Alternating moves, each player cuts out a unit square of the grid. A player loses if his/her move causes the grid to lose circular connection (two unit squares that only touch at a corner are considered to be disconnected). Suppose Mari makes the first move. Which player has a winning strategy? (Grade 10.)

Answer: Mari, if n is odd; Jüri, if n is even.

Solution 1. Consider the grid immediately before one of the players is forced to make a losing move. Divide missing squares into horizontal blocks of consecutive squares in one row of the grid. Clearly, no upper and lower block can overlap, as this would disconnect the grid. If the whole upper row is missing, then the lower row has no

missing squares, and vice versa. In this case, there have been n moves, so Mari wins if n is odd and Jüri wins if n is even.

Assume now no horizontal missing block covers the whole cylinder. Notice that if a block is missing in the upper row, then its neighbouring column cannot have a missing square in the lower row, and vice versa. Thus, there is always a full column between two missing blocks. On the other hand, if there was more than one column between some two missing blocks, it would be possible to make another move without losing by extending one of the missing blocks. Finally, if two consecutive missing blocks were both in the same row, it would be possible to make a move by removing the square between them.

We conclude that before the last move, the missing blocks are in the upper and lower row, alternatingly, implying that the number of missing blocks is even. As there is exactly one full column between two neighbouring missing blocks, the number of full columns f is also even. The remaining columns have exactly one missing square, so the number of missing squares is $m = n - f$. If n is odd, m is odd and Mari wins; if n is even, m is even and Jüri wins.

Solution 2. Consider the planes of symmetry of the cylinder that intersect the midpoints of its bases. Choose a plane p such that at least one of its intersections with the cylinder surface coincides with a grid line.

If n is even, both intersections coincide with the grid and reflecting over p divides the unit squares into symmetric pairs. Jüri's strategy is to cut out a square symmetric to the square Mari chose. We claim that if Mari's move does not disconnect the grid, then neither does Jüri's follow-up move. Indeed, if Mari removes a square not adjacent to p , then Mari's move does not affect the grid around Jüri's square. Thus, the moves are mirror images of each other and have identical impact. If Mari removes a square adjacent to p , then before Jüri's move, the grid around his square has one additional missing adjacent square compared to the grid around Mari's square before her move. Thus, Jüri's move can only disconnect the grid along the line between his square and Mari's. However, since the two removed squares are in the same row, and the two other squares in the same column must be present (for otherwise Mari's move would have disconnected the grid), the disconnect does not happen. It follows that the grid can only disconnect after Mari's move, so Jüri has a winning strategy.

If n is odd, reflecting over p divides all squares into symmetric pairs save for two squares that p intersects in the middle. Let Mari first choose one of those two squares. Then Jüri cannot choose the other square without losing, as this would disconnect the square. Thus, Mari can now use the strategy of choosing symmetric squares to win the game.

FR-9. The teacher asks Arno to choose some of the positive factors of 2009^{10} such that no chosen factor divides another chosen factor. At most how many factors can Arno choose? (*Grade 10.*)

Answer: 11.

Solution. Since $2009 = 7^2 \cdot 41$, where 7 and 41 are primes, we can represent all factors of 2009^{10} in the form $7^n \cdot 41^m$, where $0 \leq n \leq 20$ and $0 \leq m \leq 10$. Since there are 11 possible choices for m , Arno can choose at most 11 factors. Otherwise, by the Pigeonhole principle, two of the chosen factors would have the same exponent m , and the one with

a smaller exponent n would divide the other.

Now, we show that none of the 11 factors of the form $7^{20-m} \cdot 41^m$, $m = 0, 1, \dots, 10$, is divisible by another. Assume by contradiction that for some two m_1 and m_2 , $7^{20-m_1} \cdot 41^{m_1}$ divides $7^{20-m_2} \cdot 41^{m_2}$. But then $20 - m_1 \leq 20 - m_2$ and $m_1 \leq m_2$, implying $m_1 = m_2$, contradiction.

FR-10. Let $n > 18$ be a positive integer such that $n - 1$ and $n + 1$ are both primes. Prove that n has at least 8 different positive factors. (Grade 11.)

Solution 1. First, since of the three consecutive integers $n - 1, n, n + 1$, two are primes, the third, n , must be divisible by both 2 and 3. Thus, 1, 2, 3 and 6 are factors of n .

Next, $n > 18 = 3 \cdot 6$. If $n = 5 \cdot 6 = 30$, we get four additional factors 5, 10, 15 and 30. We can exclude $n = 4 \cdot 6 = 24$ and $n = 6 \cdot 6 = 36$, as $24 + 1$ and $36 - 1$ are not prime.

Finally, if $n > 6 \cdot 6$, then $6 < \frac{n}{6}$, so n has four additional factors $\frac{n}{1}, \frac{n}{2}, \frac{n}{3}, \frac{n}{6}$ larger than 6.

Solution 2. As above, note that n is divisible by 2 and 3. We can express the number of factors via the exponents in the canonical representation of n . If n has a third prime factor, then its canonical representation contains at least 3 primes with exponents at least 1, so the number of factors is at least $(1 + 1) \cdot (1 + 1) \cdot (1 + 1) = 8$. Assume now $n = 2^a 3^b$. If $a = 1$, then $b > 2$, and vice versa, so the number of different factors in this case is at least $(1 + 1) \cdot (3 + 1) = 8$. Finally, if $a, b > 1$, the number of different factors is at least $(2 + 1) \cdot (2 + 1) = 9$.

FR-11. Find all real numbers k that satisfy $0 \leq a + b - kab \leq 1$ for all real numbers a and b such that $0 \leq a \leq 1$ and $0 \leq b \leq 1$. (Grade 11.)

Answer: $1 \leq k \leq 2$.

Solution. First, we show that if $1 \leq k \leq 2$, then k satisfies the condition. Let $0 \leq a, b \leq 1$, then $a^2 \leq a$ and $b^2 \leq b$. We get

$$0 \leq (a - b)^2 = a^2 + b^2 - 2ab \leq a + b - kab \leq a + b - ab = 1 - (1 - a)(1 - b) \leq 1.$$

Next, we show that the condition does not hold for $k < 1$ and $k > 2$. Take $a = b = 1$, then $a + b - kab = 2 - k$. Now $k < 1$ yields $a - b - kab = 2 - k > 1$ and $k > 2$ yields $a - b - kab = 2 - k < 0$.

FR-12. Numbers 1 to n^2 are written in some order in the unit squares of an $n \times n$ square such that in any rectangle consisting of some of those unit squares, the sum of the numbers in two opposite corner squares equals the sum of the numbers in the other two corner squares. Find all possible values for the sum of all numbers on a diagonal of the $n \times n$ square. (Grade 11.)

Answer: $\frac{n(n^2 + 1)}{2}$.

Solution. Denote the unit square in row i , column j by (i, j) and the number written in that square by t_{ij} . Let $D = t_{11} + t_{22} + \dots + t_{nn}$ be the sum of the numbers on the main diagonal. For all $i, j = 1, 2, \dots, n$, we have $t_{ii} + t_{jj} = t_{ij} + t_{ji}$. By summing n^2 such equations over all values of i and j , we get $2nD = 2 \cdot (1 + 2 + \dots + n^2) = 2 \cdot \frac{n^2(n^2 + 1)}{2}$,

as each of the n numbers on the diagonal occurs an equal $2n$ times on the lhs, and each of the n^2 numbers occurs twice on the rhs. Solving for D gives $D = \frac{n(n^2 + 1)}{2}$.

FR-13. Call a point in a plane *rational* (resp. *irrational*) if both its coordinates are rational (resp. irrational).

- Does every point on the plane lie on some line defined by two rational points?
- Does every point on the plane lie on some line defined by two irrational points?

(Grade 11.)

Answer: a) no; b) yes.

Solution. a) Two different rational points $A(x_1, y_1)$ and $B(x_2, y_2)$ define a line given by

$$(x - x_1)(y_2 - y_1) = (y - y_1)(x_2 - x_1),$$

or $ax + by + c = 0$, where $a = y_2 - y_1$, $b = x_1 - x_2$ and $c = y_1(x_2 - x_1) - x_1(y_2 - y_1)$ are all rational.

We prove that point $(x, y) = (\sqrt{2}, \sqrt{3})$ does not satisfy $ax + by + c = 0$ for any triple of rational numbers (a, b, c) . Indeed, assume $a\sqrt{2} + b\sqrt{3} + c = 0$, then $a\sqrt{2} = -b\sqrt{3} - c$, which gives $2a^2 = 3b^2 + 2bc\sqrt{3} + c^2$, implying that $2bc\sqrt{3} = 2a^2 - 3b^2 - c^2$ is rational. The latter is only possible if $b = 0$ or $c = 0$.

If $b = 0$, then $a\sqrt{2} = -c$ is rational. This is only possible when $a = c = 0$, i.e., points A and B coincide.

If $c = 0$, then $a\sqrt{2} = -b\sqrt{3}$. Now $a = 0$ implies $b = 0$ and vice versa; in this case again, the points coincide. Thus, $a, b \neq 0$ and $-\frac{a}{b} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$. However, $\frac{a}{b}$ is rational while $\sqrt{6}$ is irrational, contradiction.

Thus, no line defined by two rational points contains point $(\sqrt{2}, \sqrt{3})$.

b) Take an arbitrary point (x, y) . Consider points $A(x + \alpha_1, y + \beta_1)$ and $B(x + \alpha_2, y + \beta_2)$, where

$$(\alpha_1, \alpha_2) = \begin{cases} (1, 2), & x \in \mathbb{I}, \\ (\sqrt{2}, 2\sqrt{2}), & x \in \mathbb{Q}, \end{cases} \quad (\beta_1, \beta_2) = \begin{cases} (1, 2), & y \in \mathbb{I}, \\ (\sqrt{2}, 2\sqrt{2}), & y \in \mathbb{Q}. \end{cases}$$

Points A and B are irrational, since each of their coordinates is a sum of a rational and an irrational number. It is also easy to verify that point (x, y) lies on line AB :

$$\frac{x - (x + \alpha_1)}{(x + \alpha_2) - (x + \alpha_1)} = \frac{-\alpha_1}{\alpha_2 - \alpha_1} = -1 = \frac{-\beta_1}{\beta_2 - \beta_1} = \frac{y - (y + \beta_1)}{(y + \beta_2) - (y + \beta_1)}.$$

FR-14. Four equally sized spheres are placed in a larger sphere such that each of the small spheres touches the remaining three small spheres and the large sphere. Is the total volume of the four small spheres equal to, larger or smaller than the remaining volume of the large sphere? (Grade 12.)

Answer: smaller.

Solution 1. Assume the total volume of the small spheres is at least one half of the volume of the large sphere. Let the radii of the small and big sphere be r and R , respectively. Since the ratio of the volumes equals the ratio of the radii in cubic power, we have $2 \cdot 4 \cdot r^3 \geq R^3$, or $r \geq \frac{R}{2}$. The strict inequality $r > \frac{R}{2}$ is clearly impossible, as the centre of the large sphere would have to lie in each of the smaller spheres simultaneously. If $r = \frac{R}{2}$, any two small spheres should touch at the centre of the large sphere (i.e., the 6 points at which the small spheres touch should all coincide), which is also impossible. We conclude that the total volume of the small spheres is less than one half of the volume of the large sphere.

Solution 2. As above, we show that the total volume of the small spheres is less than half of the volume of the large sphere. Let the radius of the small spheres be 1, then the radius of the large sphere can be written as $1 + a$, where a is the distance from the centre to the vertex in a regular tetrahedron of side length 2. Consider a triangle with vertices in the centre of this tetrahedron, and at two of its vertices (i.e., at the centres of two small spheres). The triangle has side lengths 2, a and a . The triangle inequality gives $2a = a + a > 2$, so $a > 1$.

The volume of half the large sphere is $V = \frac{4}{3} \cdot \pi \cdot (1 + a)^3$, while the total volume of the small spheres is $v = 4 \cdot \frac{4}{3} \cdot \pi \cdot 1$. As $a > 1$, $(1 + a)^3 > 8$, and $V > 2v$.

Remark. The exact value of a in Solution 2 is $\frac{\sqrt{6}}{2}$.

FR-15. Let n be a non-negative integer such that $3^n + 3^{n+1} + \dots + 3^{2n}$ is a perfect square. Prove that n is divisible by 4. (*Grade 12.*)

Solution. Adding as a sum of geometric progression, we get

$$3^n + 3^{n+1} + \dots + 3^{2n} = 3^n \cdot (1 + 3 + \dots + 3^n) = 3^n \cdot \frac{3^{n+1} - 1}{2}.$$

The factors 3^n and $\frac{3^{n+1} - 1}{2}$ are relatively prime: the former can have only 3 as its prime factor while the latter is not divisible by 3. As the product is a perfect square, both factors are perfect squares. The factor 3^n gives now that n is even.

Suppose n is not divisible by 4. As n is even, $n \equiv 2 \pmod{4}$ whence $n + 1 \equiv 3 \pmod{4}$. Note that $3^4 = 81$; from $16 \mid 80$ we get $3^4 \equiv 1 \pmod{16}$ whence $3^{n+1} \equiv 3^3 \equiv 11 \pmod{16}$ and $3^{n+1} - 1 \equiv 10 \pmod{16}$, so $\frac{3^{n+1} - 1}{2} \equiv 5 \pmod{8}$. But a perfect square is not congruent to 5 modulo 8.

Remark. From the assumption $n \equiv 2 \pmod{4}$, an analogous contradiction can also be derived using module 5.

The solutions can also be written down in easier terms, omitting the application of the geometric progression sum formula and just studying the behaviour of the sums of powers of 3 modulo 8 (or modulo 5).

FR-16. Find all real numbers a such that polynomial $x^3 + ax - 2(a + 4)$ has exactly two distinct real roots. (Grade 12.)

Answer: $-12, -3$.

Solution 1. Note that $x^3 + ax - 2(a + 4) = (x - 2) \cdot (x^2 + 2x + (a + 4))$. Hence 2 is a root of our polynomial irrespective of a . Consider two cases.

If $x = 2$ is a single root then the quadratic polynomial $x^2 + 2x + (a + 4)$ must have exactly one real root, i.e., its discriminant equals zero. Thus $4 - 4(a + 4) = 0$, giving $a = -3$.

If $x = 2$ is a double root then $x = 2$ must be a root of $x^2 + 2x + (a + 4)$, hence $2^2 + 2 \cdot 2 + (a + 4) = 0$, giving $a = -12$. As $x^2 + 2x - 8 \neq (x - 2)^2$, the other root is different from 2 indeed.

Hence the only possibilities are $a = -3$ and $a = -12$.

Solution 2. As a polynomial with real coefficients cannot have one imaginary root and other roots real, the given cubic polynomial has exactly two distinct roots, i.e., one of its roots must be a double root. This root is also a root of its derivative $3x^2 + a$. Letting $a = -3x^2$ in the original polynomial, we get the equation $-2x^3 + 6x^2 - 8 = 0$ for roots common to the polynomial and the derivative. As $-2x^3 + 6x^2 - 8 = -2(x - 2)^2(x + 1)$, the possible common roots are $x = 2$ and $x = -1$. From $3x^2 + a = 0$, we now establish the corresponding possibilities $a = -12$ and $a = -3$.

FR-17. Prove that the ratio of the lengths of the two diagonals of a parallelogram equals the ratio of its side lengths iff the angles at the intersection of the diagonals are equal to the interior angles of the parallelogram. (Grade 12.)

Solution 1. Since the diagonals of a parallelogram bisect each other, condition $\frac{|BC|}{|CD|} = \frac{|CA|}{|BD|}$ is equivalent to $\frac{|BC|}{|CD|} = \frac{|CP|}{|PD|}$ (Fig. 13). The Sine Law in triangles BCD and CPD give $\frac{|BC|}{|CD|} = \frac{\sin \angle BDC}{\sin \angle CBD}$ and $\frac{|CP|}{|PD|} = \frac{\sin \angle PDC}{\sin \angle PCD} = \frac{\sin \angle BDC}{\sin \angle PCD}$. Thus, $\frac{|BC|}{|CD|} = \frac{|CP|}{|PD|}$ iff $\sin \angle CBD = \sin \angle PCD$. From triangle BCD , we observe $\angle CBD + \angle PCD < 180^\circ$, hence $\sin \angle CBD = \sin \angle PCD$ iff $\angle CBD = \angle PCD$. Since triangles CPD and BCD share a common angle $\angle PDC = \angle BDC$, the latter condition translates to $\angle CPD = \angle BCD$.

To conclude, we have that condition $\frac{|BC|}{|CD|} = \frac{|CA|}{|BD|}$ is equivalent to $\angle CPD = \angle BCD$.

Analogously, $\frac{|DC|}{|CB|} = \frac{|CA|}{|BD|}$ is equivalent to $\angle CPB = \angle BCD$.

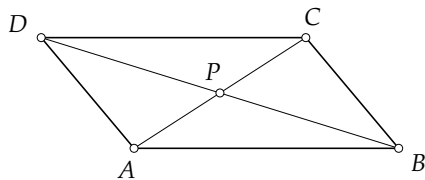


Figure 13

Solution 2. Let the diagonals AC and BD of parallelogram $ABCD$ intersect at P and let $a = |AB| = |CD|$, $b = |BC| = |AD|$ and $d = |BD|$, $e = |AC|$. We show that $\frac{a}{b} = \frac{d}{e}$ iff $\angle BCD = \angle CPD$ (analogously, we can then show that $\frac{a}{b} = \frac{e}{d}$ iff $\angle BCD = \angle CPB$).

In parallelogram $ABCD$, we have $2(a^2 + b^2) = d^2 + e^2$, or $2a^2\left(1 + \frac{b^2}{a^2}\right) = d^2\left(1 + \frac{e^2}{d^2}\right)$ (by the Cosine Law in triangles ABC and BCD). Thus, condition $\frac{a}{b} = \frac{d}{e}$ is equivalent to condition $2a^2 = d^2$. The two equations $\frac{a}{b} = \frac{d}{e}$ and $2a^2 = d^2$ are in turn equivalent to $\frac{2b}{e} = \frac{2a}{d} = \frac{d}{a}$, i.e., triangle BCD with side lengths b, a and d is similar to triangle CPD with side lengths $\frac{e}{2}, \frac{d}{2}$ and a , or $\angle BCD = \angle CPD$.

Remark. To construct a parallelogram where the ratio of diagonal lengths equals the ratio of side lengths, fix one side a and an adjacent angle α . Then the other side of the parallelogram is $b = a\sqrt{1 + \cos^2 \alpha} - a \cos \alpha$.

FR-18. Some unit squares are removed from a rectangular grid in such a way that whenever a unit square is removed, all unit squares in the rectangular area obtained by extending the left side of the unit square to the top of the grid and the bottom side of the square to the right side of the grid are also removed. Finally, a count is written in each of the remaining unit squares indicating the total number of remaining squares above that square in the same column and to the right of that square in the same row. Prove that there are at least as many even counts as odd counts written in the remaining squares. (Grade 12.)

Solution. Consider a configuration of the rectangular grid after removing some squares. We prove that if we can legally remove exactly two more adjacent squares, then after updating the counts, the number of even and odd counts both decrease by 1.

Indeed, if we can remove two squares in the same row, the columns of these squares must be empty above that row, and full below that row. In this row, each count decreases by 2, so the parities do not change. In the two columns below those two squares, each row contains one odd and an one even number (the numbers of squares remaining above are equal and the numbers of squares remaining to the right differ by one). After removing the two squares, both counts change by one, so the total number of even and odd counts remains the same. Finally, the counts in the two removed squares are 0 and 1, so removing those squares decreases both the number of even and odd counts by 1.

Analogously, removing two squares in the same column decreases the number of even and odd counts by 1.

Now, starting from the given configuration, keep removing pairs of adjacent unit squares until it is no longer possible. In the end, we must have one of two configurations:

- An empty grid. This grid contains no even or odd counts.
- A grid with $k \geq 1$ rows where the topmost row has 1 square, the next row 2 squares, etc., until the bottommost row has k squares. Then, the square in the top row has count 0, the squares in the second row have counts 2 and 0, the third row contains 4, 2 and 0, etc.

In both configurations, there are at least as many even counts as odd counts. Compared to the original configuration, we decreased the even and odd counts by an equal num-

ber, so also in the original configuration, there had to be at least as many even counts as odd counts.

IMO Team Selection Contest

First day

TS-1. For arbitrary pairwise distinct positive real numbers a, b, c , prove the inequality

$$\frac{(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3}{(a - b)^3 + (b - c)^3 + (c - a)^3} > 8abc.$$

Solution. Denote $a - b = x$ and $b - c = y$; then $c - a = -(x + y)$. For the denominator of the lhs,

$$\begin{aligned} (a - b)^3 + (b - c)^3 + (c - a)^3 &= x^3 + y^3 - (x + y)^3 = -3x^2y - 3xy^2 = \\ &= -3xy(x + y) = 3(a - b)(b - c)(c - a). \end{aligned}$$

For the numerator, analogously,

$$(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3(a^2 - b^2)(b^2 - c^2)(c^2 - a^2).$$

Thus $\frac{(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3}{(a - b)^3 + (b - c)^3 + (c - a)^3} = (a + b)(b + c)(c + a)$. Hence the given inequality is equivalent to $(a + b)(b + c)(c + a) > 8abc$. This inequality holds due to AM-GM (use $a + b \geq 2\sqrt{ab}$ and note that $a \neq b$; similarly for other pairs).

TS-2. Call a finite set of positive integers *independent* if its elements are pairwise coprime, and *nice* if the arithmetic mean of the elements of every non-empty subset of it is an integer.

- a) Prove that for any positive integer n there is an n -element set of positive integers which is both independent and nice.
- b) Is there an infinite set of positive integers whose every independent subset is nice and which has an n -element independent subset for every positive integer n ?

Answer. b) No.

Solution 1. a) Let $A = \{n! + 1, 2 \cdot n! + 1, \dots, n \cdot n! + 1\}$. For any two elements $k \cdot n! + 1$ and $l \cdot n! + 1$ where $1 \leq k < l \leq n$,

$$\begin{aligned} \gcd(k \cdot n! + 1, l \cdot n! + 1) &= \gcd(k \cdot n! + 1, (l \cdot n! + 1) - (k \cdot n! + 1)) \\ &= \gcd(k \cdot n! + 1, (l - k)n!). \end{aligned}$$

The number $k \cdot n! + 1$ is not divisible by any of $2, \dots, n$, thus by none of their prime divisors; the only prime divisors of $(l - k)n!$ are those of $2, \dots, n$. So $\gcd(k \cdot n! + 1, l \cdot n! + 1) = 1$, thus A is independent.

From arbitrary m elements $k_1 \cdot n! + 1, k_2 \cdot n! + 1, \dots, k_m \cdot n! + 1$ of A where $1 \leq m \leq n$, we get

$$\begin{aligned} (k_1 \cdot n! + 1) + (k_2 \cdot n! + 1) + \dots + (k_m \cdot n! + 1) &= (k_1 + k_2 + \dots + k_m)n! + m \\ &= m \cdot \left((k_1 + k_2 + \dots + k_m) \cdot \frac{n!}{m} + 1 \right). \end{aligned}$$

Since $\frac{n!}{m}$ is an integer, the sum of the m selected elements is divisible by m , so their arithmetic mean is an integer. Hence A is nice.

b) Note that the difference of any two elements of an n -element nice set is divisible by every positive integer less than n . Indeed, let X be an n -element nice set. Fix an integer i , $2 \leq i \leq n-1$, and arbitrary $a, b \in X$. If s is the sum of some $i-1$ elements of X different from a and b then both $a+s$ and $b+s$ are divisible by i , thus so is their difference $a-b$. Assume there is an infinite set B of positive integers such that for every n there is n -element independent subset and that every independent subset of B is nice. We show that every finite independent subset A can be extended to larger independent subsets by adding new numbers. Indeed, there is only a finite number, say k , of primes which divide some element of A . By Pigeonhole principle, a $k+1$ -element independent subset of B has an element that is coprime with all elements of A , and we can add this element to A to form a larger independent set.

Let a and b be different coprime elements of B (they exist since there is a 2-element independent subset of B) and consider an infinite subset of B containing both a and b and having pairwise coprime elements (it can be constructed by infinitely repeating the steps described above starting from $A = \{a, b\}$). Since any of its finite subsets containing a and b is nice, $a-b$ is divisible by every positive integer. So $a=b$ and we obtained a contradiction showing there is no such set B .

Solution 2. a) Use *Dirichlet's Theorem*: for any two coprime numbers a and d there are infinitely many primes of the form $a+kd$ where $k \geq 0$. Choosing $a=1$ and $d=n!$ we have infinitely many primes of the form $kn!+1$ where $k \geq 0$. Choosing some n of them we have an independent set A . Similarly to Solution 1 we establish the niceness of A .

b) Let B be an infinite set of positive integers such that for every n there is n -element independent subset and that every independent subset of B is nice. Let $a \in B$. Let C be any independent subset of B of size greater than a . As established in Solution 1, the difference of any two elements of B is divisible by a . So all elements of B are congruent modulo a and thus have the same greatest common divisor with a . Since B is independent, it has to be 1.

We have that all elements of each independent subset of B of size greater than a are coprime with a . So we can add a to any such set to form an independent set with more than a elements containing a . So also a has to be coprime with a , thus $a=1$. Since the choice of a was arbitrary, $B = \{1\}$ which is impossible.

TS-3. Find all natural numbers n for which there exists a convex polyhedron satisfying the following conditions:

- (i) Each face is a regular polygon.
- (ii) Among the faces, there are polygons with at most two different numbers of edges.

(iii) There are two faces with common edge that are both n -gons.

Answer: 3, 4, 5, 6, 8, 10.

Solution. For $n = 3, 4, 5$, the well-known regular polyhedrons satisfy the conditions. If we remove a regular pyramid around each vertex of a regular tetrahedron, cube, or dodekahedron, the faces of these polyhedrons become 6-gons, 8-gons and 10-gons, respectively. Clearly one can make the cuts so that all these faces are regular polygons. At place of vertices that are cut off, equilateral triangles appear. Hence also 6, 8, 10 satisfy the conditions of the problem.

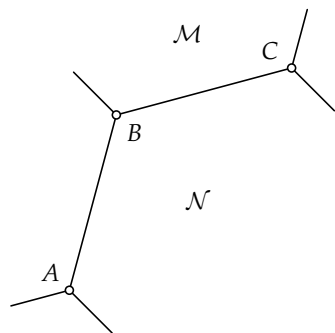


Figure 14

Show now that there are no other answers. Suppose we have a polyhedron satisfying the conditions for some n . Consider two neighbouring n -gons; let AB be the edge separating them (Fig. 14). Suppose A belongs to exactly k other faces. Then $2\pi > 2 \cdot \frac{(n-2)\pi}{n} + k \cdot \frac{\pi}{3}$ or, equivalently, $n < \frac{12}{k}$. Thus $k \geq 2$ implies $n < 6$ and $k = 1$ implies $n < 12$.

All we have to show is that $n = 7, 9, 11$ are impossible; so assume n is odd and $6 < n < 12$. Then vertex A , as well as vertex B , belongs to exactly one face other than the two n -gons. Let m be the number of vertices of these faces.

Let \mathcal{N} be one of the two n -gons under consideration and let \mathcal{M} be the face meeting them at B . Let C be the other endpoint of the edge separating \mathcal{M} and \mathcal{N} . Let γ be the sum of all interior angles of the faces other than \mathcal{M} and \mathcal{N} that meet at C . Then clearly $\gamma \geq \frac{(n-2)\pi}{n}$ (if only one face meets \mathcal{M} and \mathcal{N} at C then it must be an n -gon, and if there are more faces then the sum of their angles at C is greater). We obtain the inequalities

$$\pi > \frac{20}{21}\pi = 2\pi - \frac{\pi}{3} - \frac{5\pi}{7} \geq 2\pi - \frac{(m-2)\pi}{m} - \frac{(n-2)\pi}{n} > \gamma \geq \frac{(n-2)\pi}{n} \geq \frac{5\pi}{7}.$$

From $\pi > \gamma$ we see that there are at most 2 other faces meeting \mathcal{M} and \mathcal{N} at C . If there were 2 of them, they both would have no more than 5 vertices; therefore they would not be n -gons and thus they both would be polygons of the second kind. But for this situation, $\pi > \gamma$ gives that they both must be triangles which is contradictory to $\gamma \geq \frac{5\pi}{7}$. Consequently, there is only one face meeting \mathcal{M} and \mathcal{N} at C and this must be an n -gon.

Continuing the same way, we find that the faces surrounding \mathcal{N} are alternately n -gons and m -gons. As n is odd by assumption, we get $n = m$. Thus three n -gons meet at A , implying $n < 6$. This completes the solution.

Remark. The condition (iii) of the problem is necessary. Without it, a required polyhedron would exist for arbitrary n : a right prism whose bases are regular n -gons and lateral faces are squares would do.

Second day

TS-4. Points A' , B' , C' are chosen on the sides BC , CA , AB of triangle ABC , respectively, so that $\frac{|BA'|}{|A'C|} = \frac{|CB'|}{|B'A|} = \frac{|AC'|}{|C'B|}$. The line which is parallel to line $B'C'$ and goes through point A' intersects the lines AC and AB at P and Q , respectively. Prove that $\frac{|PQ|}{|B'C'|} \geq 2$.

Solution 1. Denote $\frac{|BA'|}{|A'C|} = \frac{|CB'|}{|B'A|} = \frac{|AC'|}{|C'B|} = k$.

If $k = 1$ then $B'C'$ is the segment connecting midpoints of sides AB and AC of the triangle ABC , so it is parallel to the side BC . Thus the lines PQ and BC coincide, meaning $P = C$ and $Q = B$. Thus $\frac{|PQ|}{|B'C'|} = \frac{|BC|}{|B'C'|} = 2$.

Now assume that $k \neq 1$ (Fig. 15). Let B'' be the point on the side AC such that $\frac{|AB''|}{|B''C|} = k$ (in other words, B'' is symmetric to B' w.r.t. the perpendicular bisector of AC).

By the intercept theorem, $B''C' \parallel BC$ and $A'B'' \parallel AB$. Thus $BC'B''A'$ is a parallelogram; also $\triangle PA'C \sim \triangle B'C'B''$ since the corresponding sides are collinear (the triangles exist since $k \neq 1$). Hence $\frac{|A'P|}{|B'C'|} = \frac{|A'C|}{|C'B''|} = \frac{|A'C|}{|BA'|} = \frac{1}{k}$. Let C'' be a point on the side AB

such that $\frac{|BC''|}{|C''A|} = k$. By analogy, we have that $\frac{|A'Q|}{|B'C'|} = \frac{|A'B|}{|CA'|} = k$. It follows that $\frac{|PQ|}{|B'C'|} = \frac{|A'P| + |A'Q|}{|B'C'|} = \frac{1}{k} + k > 2$.

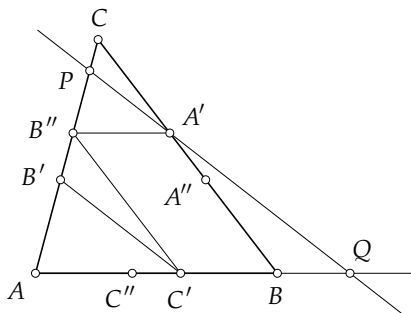


Figure 15

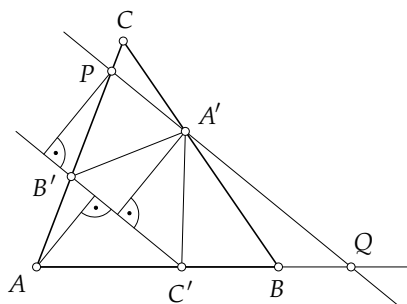


Figure 16

Solution 2. From the intercept theorem,

$$\frac{|PQ|}{|B'C'|} = \frac{|PA|}{|AB'|} = \frac{|PB'| + |AB'|}{|AB'|} = \frac{|PB'|}{|AB'|} + 1.$$

Let $d(X, l)$ be the distance of the point X from the line l and S_{Δ} the area of the triangle Δ . Considering lines perpendicular to $B'C'$ through points P , resp. A (Fig. 16), we have

$$\frac{|PB'|}{|AB'|} = \frac{d(P, B'C')}{d(A, B'C')} = \frac{d(A', B'C')}{d(A, B'C')} = \frac{S_{\Delta A'B'C'}}{S_{\Delta AB'C'}}.$$

Thus

$$\frac{|PQ|}{|B'C'|} \geq 2 \iff \frac{|PB'|}{|AB'|} \geq 1 \iff \frac{S_{\Delta A'B'C'}}{S_{\Delta AB'C'}} \geq 1 \iff S_{\Delta A'B'C'} \geq S_{\Delta AB'C'}.$$

We show now that indeed $S_{\Delta A'B'C'} \geq S_{\Delta AB'C'}$. As $\frac{|BA'|}{|A'C|} = \frac{|CB'|}{|B'A|} = \frac{|AC'|}{|C'B|}$, we have

$\frac{|BA'|}{|BC|} = \frac{|CB'|}{|CA|} = \frac{|AC'|}{|AB|} = x$ where $0 < x < 1$. We obtain

$$\begin{aligned} S_{\Delta AB'C'} &= \frac{1}{2} \cdot |AC'| \cdot d(B', AB) = \frac{1}{2} \cdot (x \cdot |AB|) \cdot ((1-x) \cdot d(C, AB)) \\ &= x(1-x) \cdot \frac{1}{2} \cdot |AB| \cdot d(C, AB) = x(1-x) \cdot S_{\Delta ABC}. \end{aligned}$$

Since from AM-GM, $x(1-x) \leq \left(\frac{x+(1-x)}{2}\right)^2 = \frac{1}{4}$, we have $S_{\Delta AB'C'} \leq \frac{1}{4} \cdot S_{\Delta ABC}$.

Analogously $S_{\Delta A'BC'} \leq \frac{1}{4} \cdot S_{\Delta ABC}$ and $S_{\Delta A'B'C} \leq \frac{1}{4} \cdot S_{\Delta ABC}$. To summarize,

$$\begin{aligned} S_{\Delta A'B'C'} &= S_{\Delta ABC} - (S_{\Delta AB'C'} + S_{\Delta A'BC'} + S_{\Delta A'B'C}) \\ &\geq S_{\Delta ABC} - \frac{3}{4} \cdot S_{\Delta ABC} = \frac{1}{4} \cdot S_{\Delta ABC} \geq S_{\Delta AB'C'}. \end{aligned}$$

Solution 3. As $\frac{|BA'|}{|A'C|} = \frac{|CB'|}{|B'A|} = \frac{|AC'|}{|C'B|}$, we also have $\frac{|BA'|}{|BC|} = \frac{|CB'|}{|CA|} = \frac{|AC'|}{|AB|}$, let this ratio be x . Then

$$\begin{aligned} \overrightarrow{AC'} &= x \cdot \overrightarrow{AB}, & \overrightarrow{AB'} &= (1-x) \cdot \overrightarrow{AC}, \\ \overrightarrow{AA'} &= \overrightarrow{AB} + \overrightarrow{BA'} = \overrightarrow{AB} + x \cdot \overrightarrow{BC} = \\ &= \overrightarrow{AB} + x \cdot (\overrightarrow{AC} - \overrightarrow{AB}) = (1-x) \cdot \overrightarrow{AB} + x \cdot \overrightarrow{AC}. \end{aligned}$$

Let $\frac{|PQ|}{|B'C'|} = v$; we need to show that $v \geq 2$. Since $PQ \parallel B'C'$, we obtain $\frac{|AP|}{|AB'|} = \frac{|AQ|}{|AC'|} = v$. Hence $\overrightarrow{AP} = v \cdot \overrightarrow{AB'} = v \cdot (1-x) \cdot \overrightarrow{AC}$ and $\overrightarrow{AQ} = v \cdot \overrightarrow{AC'} = v \cdot x \cdot \overrightarrow{AB}$. As P, A', Q are collinear, we have $\overrightarrow{PA'} = z \cdot \overrightarrow{PQ}$ or $\overrightarrow{AA'} - \overrightarrow{AP} = z \cdot (\overrightarrow{AQ} - \overrightarrow{AP})$. Substituting the formulae for $\overrightarrow{AA'}, \overrightarrow{AP}, \overrightarrow{AQ}$, we have

$$(1-x) \cdot \overrightarrow{AB} + x \cdot \overrightarrow{AC} - v \cdot (1-x) \cdot \overrightarrow{AC} = z \cdot v \cdot x \cdot \overrightarrow{AB} - z \cdot v \cdot (1-x) \cdot \overrightarrow{AC}.$$

Since \overrightarrow{AB} and \overrightarrow{AC} are not collinear, the identity can only hold if their coefficients on both sides are the same. We have the system of equations

$$\begin{cases} 1-x = zvx, \\ x - v(1-x) = -zv(1-x). \end{cases}$$

From the first equation $zv = \frac{1-x}{x}$, and substituting this into the second we get $x - v(1-x) = -\frac{1-x}{x} \cdot (1-x)$. Thus $v = \frac{\frac{(1-x)^2}{x} + x}{1-x} = \frac{(1-x)^2 + x^2}{(1-x)x}$. Since $(1-x)^2 + x^2 \geq 2(1-x)x$, the last equation gives $v \geq 2$, just as required.

Solution 4. Let $\frac{|PQ|}{|B'C'|} = v$; we have to show $v \geq 2$. As $PQ \parallel B'C'$, we have $\frac{|AP|}{|AB'|} = \frac{|AQ|}{|AC'|} = v$. W.l.o.g., assume $|AP| \geq |AC|$ and $|AQ| \leq |AB|$. Then

$$\frac{|AQ|}{|QB|} = \frac{v \cdot |AC'|}{|QB|} = \frac{v \cdot |AC'|}{|AB| - |AQ|} = \frac{v \cdot |AC'|}{|AB| - v \cdot |AC'|} = \frac{v}{\frac{|AB|}{|AC'|} - v} = \frac{v}{\frac{1+k}{k} - v}$$

and, analogously,

$$\frac{|CP|}{|PA|} = \frac{|CP|}{v \cdot |AB'|} = \frac{|AP| - |AC|}{v \cdot |AB'|} = \frac{v \cdot |AB'| - |AC|}{v \cdot |AB'|} = 1 - \frac{|AC|}{v \cdot |AB'|} = 1 - \frac{1+k}{v}$$

From Menelaus' theorem, $\frac{|AQ|}{|QB|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CP|}{|PA|} = 1$. Now substituting gives $\frac{v}{\frac{1+k}{k} - v} \cdot k \cdot \frac{v - (1+k)}{v} = 1$ which is equivalent to $k v - k(1+k) = \frac{1+k}{k} - v$. Hence $v(1+k) = \frac{1}{k}(1+k) + k(1+k)$, leading to $v = \frac{1}{k} + k$. Therefore $v \geq 2$.

TS-5. A strip consists of n squares which are numerated in their order by integers $1, 2, 3, \dots, n$. In the beginning, one square is empty while each remaining square contains one piece. Whenever a square contains a piece and its some neighbouring square contains another piece while the square immediately following the neighbouring square is empty, one may raise the first piece over the second one to the empty square, removing the second piece from the strip.

Find all possibilities which square can be initially empty, if it is possible to reach a state where the strip contains only one piece and a) $n = 2008$; b) $n = 2009$.

Answer: a) 2, 5, 2004, 2007; b) none.

Solution. Interpret the move so that the piece at distance 2 from the empty square is removed from the strip and the piece from the intermediate square moves to the empty square. In the following, assume $n \geq 5$.

First show that, under this interpretation, the location of any piece, as long as it is on the board, does not change more than by one square. For that, we show that after a move in some direction, no piece can make next move in the same direction. Perform induction on the distance between the piece and the end of the strip from which the piece went farther on the move – assume that the claim holds for smaller distances. Clearly there are two possibilities for the state after the move: (1) there are no pieces between the piece that moved and the end of the strip under consideration; (2) there are other pieces in that area but the nearest among them is separated from the piece that moved by at

least two empty squares. But for moving in the same direction, there must be a piece on the square where the piece came from. By the induction hypothesis, the pieces between the piece that moved and the end of the strip under consideration do not get farther by more than one square. Hence the piece that moved cannot move further in the same direction.

Note that whenever we divide the squares into two non-empty groups then, in order to achieve the desired final state, one must first obtain a state where some pieces from different groups are on neighbouring squares – otherwise one could not do a move that takes off the last piece of one group. Thus if there are three consecutive empty squares, in both sides of which there are pieces, then the required final state is impossible to achieve because the pieces from the two groups do not move nearer to each other by more than one square and there always remains an empty square between them.

Analogously, if there is a lonely piece separated by at least two empty squares from the nearest piece then the required final state is impossible to achieve since the other pieces do not reach its neighbouring square and it cannot move itself.

Independently on the location of the empty square, the state after the first move of the game has two consecutive empty squares and other squares containing a piece. Call these two squares *central* (Fig. 17). We show that, from this state, it is possible to reach the desired final state if and only if there are 2 pieces at one side and an even number of pieces at the other side of the central squares.

Suppose that this condition is fulfilled; w.l.o.g. there are 2 pieces on the left and $2k$ pieces on the right. Show by induction on k that the desired final state can be achieved. If $k = 0$ then there are only 2 pieces on the board and they are on consecutive squares, so one move reaches the goal. If $k > 0$ then first move with the piece on square 2 to the right and with the piece on square 5 to the left. In the resulting state, pieces are on squares 3 and 4, the following two squares are empty and $2(k - 1)$ consecutive squares contain pieces. By the induction hypothesis (it holds for the substrip starting from square 3) it is possible to achieve the desired final state from this state.

Suppose now there being more than 2 pieces on both sides of the central squares. In order to achieve the desired final state, the pieces from both sides must get together. It is possible only if the rightmost piece of the left-hand side moves to the right and the leftmost piece of the right-hand side moves to the left. The pieces immediately following these pieces are removed and the remaining pieces do not move nearer by more than one square. Thus an empty square remains on both sides of the central squares where no remaining pieces can ever come. Hence when one of the pieces on the central squares makes a move, the other one is removed and three consecutive squares become empty, whereby there are still pieces on both sides from them. Thus the desired final state cannot be achieved.

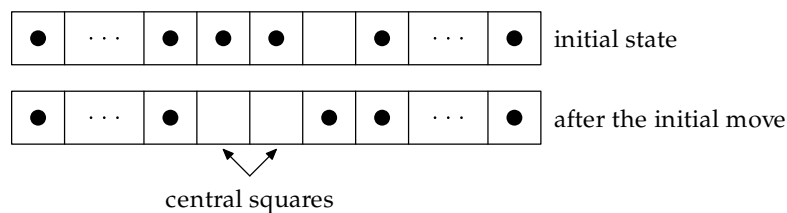


Figure 17

Also if there is 1 piece on one side from the central squares then the desired final state is unreachable. If there is 0 pieces on one side then there will be one piece after one move, consequently the final state is unreachable again. It remains to study the case with 2 pieces on one side – w.l.o.g., on the left – and an odd number $2k + 1$ of pieces on the other side. Show by induction on k that the required final state is unreachable even if the strip had more empty squares to the left from square 1. If $k = 0$ then both possible moves (from 2 to the right and from 1 to the left) leads to a stub configuration with 2 lonely pieces. Let $k > 0$. The move from 1 to the left gives a state where three consecutive squares are empty, thus uninteresting. The moves from 2 to the right and from 5 to the left remain, which both are certainly performed during the play. Suppose w.l.o.g. that the very first move is from 2 to the right because the possible moves before it concern the pieces on the right only and they can be performed also if the move from 2 to the right is already done. Hence in the state after two moves, squares 3 and 4 contain pieces, the next two squares are empty and then $2(k - 1) + 1$ consecutive squares with pieces follow. By the induction hypothesis, the desired state is unreachable.

We have got that the desired state is possible only for even n since only then there can be 2 pieces on one side and an even number of pieces on the other side from two central squares. In the case of even n , the state where squares 3 and 4 are empty can arise if, initially, 2 or 5 is empty. Analogously, the symmetric case is handled. Hence, for $n = 2008$, the squares 2, 5, 2004, 2007 can be initially empty.

TS-6. For any positive integer n , let $c(n)$ be the largest divisor of n not greater than \sqrt{n} and let $s(n)$ be the least integer x such that $n < x$ and the product nx is divisible by an integer y where $n < y < x$. Prove that, for every n , $s(n) = (c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1\right)$.

Solution. Take $y = c(n) \cdot \left(\frac{n}{c(n)} + 1\right)$. Then

$$n = c(n) \cdot \frac{n}{c(n)} < y < (c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1\right)$$

while

$$y \mid c(n) \cdot \frac{n}{c(n)} \cdot (c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1\right) = n \cdot \left((c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1\right)\right).$$

This implies $s(n) \leq (c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1\right)$.

For establishing minimality, choose x such that $n < x < (c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1\right)$. It has to be proven that no integer strictly between n and x divides nx .

Let y be an arbitrary integer strictly between n and x . Let $d = \gcd(y, n)$ and $y' = \frac{y}{d}$, $n' = \frac{n}{d}$.

As $c(n)$ and $\frac{n}{c(n)}$ are the central divisors of n , we have

$$c(n) + \frac{n}{c(n)} \leq d + \frac{n}{d}. \quad (4)$$

(Indeed, consider function $f(z) = z + \frac{1}{z}$; it is decreasing between 0 and 1. Then $c(n) + \frac{n}{c(n)} = \sqrt{n} \cdot f(\frac{c(n)}{\sqrt{n}})$ and $d + \frac{n}{d} = \sqrt{n} \cdot f(\frac{\min(d, \frac{n}{d})}{\sqrt{n}})$ as well as $\frac{\min(d, \frac{n}{d})}{\sqrt{n}} \leq \frac{c(n)}{\sqrt{n}} \leq 1$.) Adding $n + 1$ to both sides of (4) and factorizing, we get

$$(c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1 \right) \leq (d + 1) \cdot \left(\frac{n}{d} + 1 \right). \quad (5)$$

But $d \mid y$ implies $d \mid y - n$ and furthermore $d \leq y - n$. Thus $n + d \leq y$, implying

$$\frac{n}{d} + 1 \leq \frac{y}{d} = y'. \quad (6)$$

Inequalities (5) and (6) together give

$$dy' = y < x < (c(n) + 1) \cdot \left(\frac{n}{c(n)} + 1 \right) \leq (d + 1) \cdot \left(\frac{n}{d} + 1 \right) \leq (d + 1)y'.$$

Hence x is strictly between two consecutive multiples dy' and $(d + 1)y'$ of y' , therefore $y' \nmid x$.

If $y \mid nx$ were the case then reducing by d would lead to $y' \mid n'x$. As y' is relatively prime to n' , this would imply $y' \mid x$. This contradiction completes the solution.

Remark. The table for small values of functions c and s is

n	1	2	3	4	5	6	7	8	9	10	11	12
$c(n)$	1	1	1	2	1	2	1	2	3	2	1	3
$s(n)$	4	6	8	9	12	12	16	15	16	18	24	20
n	13	14	15	16	17	18	19	20	21	22	23	24
$c(n)$	1	2	3	4	1	3	1	4	3	2	1	4
$s(n)$	28	24	24	25	36	28	40	30	32	36	48	35

Problems Listed by Topic

Number theory: OC-2, OC-6, FR-2, FR-4, FR-9, FR-10, FR-15, TS-2, TS-6

Algebra: OC-8, FR-5, FR-6, FR-11, FR-13, FR-16, TS-1

Geometry: OC-1, OC-3, OC-7, FR-1, FR-7, FR-14, FR-17, TS-3, TS-4

Discrete mathematics: OC-4, OC-5, FR-3, FR-8, FR-12, FR-18, TS-5