Estonian Math Competitions

2004/2005

As rays BK and CL are opposite-directed, we have

$$|BC| = \frac{|AB| + |AC|}{2} \iff |BC| = |BK| + |CL| \iff K = L.$$

3. On some square of an infinite squared plane, there is a cube which covers the square exactly. The top face of the cube is white, the other faces are black. With one step, one can turn the cube over any edge so that it starts covering a neighbouring square. Is it possible to achieve a situation where the cube lies on the initial square with the white face at the bottom, making exactly
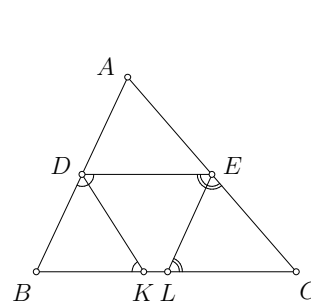
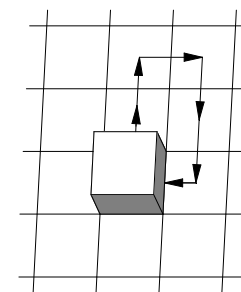a) 2004 steps;   b) 2005 steps?   *(Juniors.)*
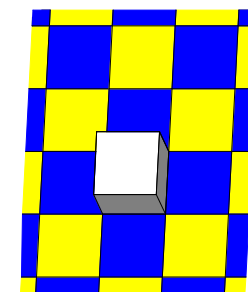
Figure 1   Figure 2   Figure 3

Answer: a) yes; b) no.

Solution 1. a) Turn the cube two steps forward, one step to the right, two steps backward, one step to the left (see Figure 2). After these six steps, the cube gets back onto the initial square but the white face is now at the bottom. The rest 1998 steps are made in pairs: turn the cube onto arbitrary neighbouring square and then turn it back onto the initial.

b) We colour squares dark and light by diagonals so that the cube lies on a light-coloured square at the beginning (see Figure 3). Since, from any square, the cube can move only to squares of the opposite colour, the cube lies on a light-coloured square after any even number of steps and on a dark-coloured square after any odd number of steps. Thus after 2005 steps, the cube lies on a square different from the initial.

Solution 2. a) Turn the cube two steps forward, one step to the left, two steps forward, two steps to the right, four steps backward, one step to the left. After these 12 steps, the cube is back on the initial square but the white face is now at the bottom. Repeating this cycle, we see that, after any odd number of repetitions, the white face of the cube is at the bottom and, after any even number of repetitions, the white face is at the top. Since $2004 = 12 \cdot 167$, we obtain the desired result by repeating this cycle for 167 times.

b) Assume that the cube makes a circuit and gets back onto the initial square. Let a, b, c, and d be the numbers of steps made during this circuit respectively to the right, to the left, up, and down. Then $a = b$ and $c = d$. Thus the cube makes altogether

Selected Problems from Open Contests

1. In the final tournament of football championship, the teams are divided into groups of four. Each team plays one game with every other team in its group. A win gives 3 points, a draw 1 point and a loss 0 points. From each group, two teams advance so that each advancing team gets at least as many points as each non-advancing team.

a) What is the smallest possible score of an advancing team?

b) What is the biggest possible score of a non-advancing team?   *(Juniors.)*

Answer: a) 2; b) 6.

Solution. a) If one team in a group wins all matches and the other three draw all matches, then the second advancing team collects just 2 points. Let us prove that a team cannot advance with less than 2 points. Indeed, if team A gets at most 1 point, it must lose at least two games. Therefore two teams get at least 3 points and outperform A, so that A cannot be in top two.

b) If one team in a group loses all matches and the other three win cyclically (X wins Y, Y wins Z and Z wins X), then three teams collect 6 points and one of them does not advance. Let us prove that a team with more than 6 points must advance. If team C gets at least 7 points, it must win at least two games. As the losing teams can get at most 6 points, team C must be in top two.

2. In triangle ABC let D, E be the midpoints of AB and AC, respectively. Prove that the intersection point of the bisectors of angles BDE and CED lies on AB if and only if the length of BC is equal to the arithmetic mean of the lengths of AB and AC. *(Juniors.)*

Solution. Let K and L be the intersection points of BC with the bisectors of angles BDE and CED, respectively (see Figure 1). As $DE \parallel BC$, we have $\angle BDK = \angle EDK = \angle BKD$, and $|BK| = |BD| = \dfrac{|AB|}{2}$. Similarily $|CL| = \dfrac{|AC|}{2}$. Therefore

$$|BK| + |CL| = \frac{|AB| + |AC|}{2}.$$

$a + b + c + d = 2(a + c)$ steps which is even number. Hence the cube cannot get back to the initial square after 2005 steps.

4. Relatively prime positive integers a and b are chosen in such a way that $\dfrac{a+b}{a-b}$ is also a positive integer. Prove that at least one of the numbers $ab + 1$ and $4ab + 1$ is a perfect square. *(Juniors.)*

Solution 1. Let $\dfrac{a+b}{a-b} = m$. Then $a + b = ma - mb$ implying $\dfrac{a}{b} = \dfrac{m+1}{m-1}$. As a and b are relatively prime, there exists an integer k such that $m + 1 = ka$ and $m - 1 = kb$. By multiplying these equalities, we get $m^2 - 1 = k^2 ab$ implying $k^2 ab + 1 = m^2$. Number k as a common divisor of numbers $m - 1$ and $m + 1$ must be a divisor of their difference 2. Hence k can only be 1 or 2 and we are done.

Solution 2. As

$$\frac{a+b}{a-b} = \frac{a-b+b+b}{a-b} = 1 + \frac{2b}{a-b},$$

we see that $\dfrac{2b}{a-b}$ must be an integer. Numbers b and $a - b$ are relatively prime because a and b are relatively prime. Hence 2 must be divisible by $a - b$. Therefore $a - b = 1$ or $a - b = 2$. The former case implies $4ab + 1 = 4(b+1)b + 1 = (2b+1)^2$, the latter case implies $ab + 1 = (b+2)b + 1 = (b+1)^2$.

5. The teacher has chosen positive integers a and b such that $\dfrac{a}{b} \cdot \sqrt{a^2 + b^2}$ is an integer.
a) Silly-Sam claims that a is divisible by every prime factor of b. Prove that he is right.
b) Silly-Sam claims that actually $b \leqslant a$. Is he right this time?   *(Seniors.)*

Answer: b) no.
Solution. a) Let p be an arbitrary prime factor of b. If the observed expression is an integer, the number $a\sqrt{a^2 + b^2}$ must be divisible by p. As p is prime, either a is divisible by p or $\sqrt{a^2 + b^2}$ is divisible by p. In the latter case, squaring gives that $a^2 + b^2$ is divisible by p^2. By the initial assumption, b^2 is divisible by p^2, hence also a^2 is divisible by p^2. Therefore a is divisible by p in both cases.

b) The teacher may choose $a = 12$ and $b = 16$. In this case, $\dfrac{a}{b} \cdot \sqrt{a^2 + b^2} = 15$ is an integer. Therefore the inequality $b \leqslant a$ might be wrong.

6. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centres O_1 and O_2, respectively, are touching externally at P. On their common tangent at P, point A is chosen, rays drawn from which touch the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at points P_1 and P_2 both different from P. It is known that $\angle P_1 A P_2 = 120°$ and angles $P_1 A P$ and $P_2 A P$ are both acute. Rays $A P_1$ and $A P_2$ intersect line $O_1 O_2$ at points G_1 and G_2, respectively. The second intersection between ray $A O_1$ and $\mathcal{C}_1$ is H_1, the second intersection between ray $A O_2$ and $\mathcal{C}_2$ is H_2. Lines $G_1 H_1$ and AP intersect at K. Prove that if $G_1 K$ is a tangent to circle $\mathcal{C}_1$, then line $G_2 K$ is tangent to circle $\mathcal{C}_2$ with tangency point H_2. *(Seniors.)*

Solution. Let $\angle O_1 A G_1 = \alpha$ and $\angle O_1 G_1 A = \beta$ (see Figure 4). If line $G_1 K$ touches circle $\mathcal{C}_1$, then $\angle A H_1 G_1 = 90°$ and $\angle H_1 G_1 O_1 = \angle A G_1 O_1 = \beta$. Also, $\angle G_1 PA = 90°$ and $\angle PA O_1 = G_1 A O_1 = \alpha$. From the right-angled triangles $A H_1 G_1$ and $G_1 PA$, we get
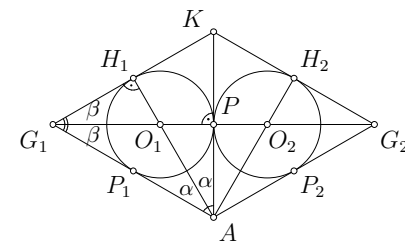
$$\alpha + 2\beta = 90°,$$
$$\beta + 2\alpha = 90°.$$

Figure 4

Solving the system gives $\alpha = \beta = 30°$. Therefore $\angle PA G_1 = 60°$ and $\angle PA G_2 = 120° - 60° = 60°$. Thus we have $\angle PA G_1 = \angle PA G_2$. As circles $\mathcal{C}_1$ and $\mathcal{C}_2$ lie in equal angles and touch at P, their radii must be equal. Therefore AK is the symmetry axis. By symmetry, line $G_2 K$ is a tangent to $\mathcal{C}_2$ and touches the circle at H_2.

7. A king wants to connect n towns of his kingdom with one-directional airways so that, from each town, exactly two airlines depart. From each town, it must be possible to fly to every other town with at most one change. Find the biggest n for which this plan is feasible. *(Seniors.)*

Answer: 6.
Solution. From a fixed town, one can get directly to two towns and further to at most four more towns. Thus the number of towns cannot exceed $1 + 2 + 4 = 7$.

Let us assume that a suitable airway plan for 7 towns exists. From each town, there must be a unique way to every other town (either direct or with one change), otherwise some town would have less than 6 possible final destinations. Without loss of generality assume that there is a direct flight from town L_1 to towns L_2 and L_3, from town L_2 to towns L_4 and L_5, and from town L_3 to towns L_6 and L_7 (see Figure 5).
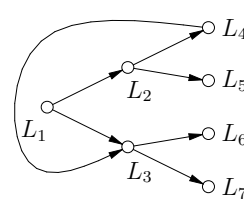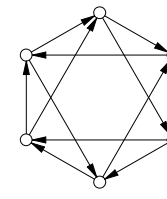

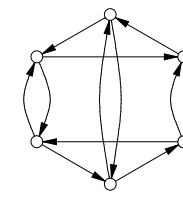Figure 5   Figure 6   Figure 7

From L_2, there must be a way to all towns in list L_1, L_3, L_6 and L_7. As the direct flights from L_2 take to towns L_4 and L_5, there must be a direct flight from L_4 to two towns in the list and from L_5 to the other two. Without loss of generality assume that there is a direct flight from L_4 to L_3. But now there can be a flight from L_4 to none of L_1, L_6, L_7 because otherwise there would be two ways to get from town L_4 to town L_3, L_6, L_7, respectively.

A suitable airway plan for 6 towns exists, as shown on Figure 6 or Figure 7.

8. For which integers a does there exist two different finite sequences of positive integers $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_l$ such that

$$(a^{i_1} + 1)(a^{i_2} + 1) \cdots (a^{i_k} + 1) = (a^{j_1} + 1)(a^{j_2} + 1) \cdots (a^{j_l} + 1)?$$

(Seniors.)

Answer: $-1, 0$, and 1.

Solution. In the case $a = -1$, both sides of the equation equal to zero whenever both sequences contain an odd number. In the case $a = 0$, both sides equal to one irrespective of the sequences. In the case $a = 1$, all factors of the products are equal to 2, so the products are equal whenever the sequences have the same length. Therefore suitable sequences exist in these three cases.

Let us prove that if $|a| > 1$, then such sequences do not exist. Suppose the contrary, i.e. $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$ are different sequences that lead to equal products. We may assume that no integer is in both sequences or else the respective terms can be cancelled in the products. After deletions, both sequences are still nonempty or else we get an equation between 1 and the product of integers not being equal to 1. We can also assume that $i_1 < j_1$.

Multiplying and then removing the parentheses on both sides gives an equation between sums of powers of a. Both sides contain term 1 and we can reduce that. This ends up in the equation of form

$$a^{i_1 + \cdots + i_k} + \cdots + a^{i_1} = a^{j_1 + \cdots + j_l} + \cdots + a^{j_1}.$$

The smallest exponent is i_1 on the left-hand side and j_1 on the right-hand side. As $j_1 \geqslant i_1 + 1$, the right-hand side is divisible by a^{i_1+1}. On the left-hand side, all terms except the last one are divisible by a^{i_1+1}. Hence the left-hand side is not divisible by a^{i_1+1}, a contradiction.

Comment. From the solution, we get that for each a ($|a| > 1$), any finite subset I of natural numbers can be assigned a unique number

$$f(I) = \prod_{i \in I}(a^i + 1).$$

The number determines the subset uniquely. This can be used in proving that the number of finite subsets of natural numbers is countable.

9. Mother has baked a platecake and cut it into $m \times n$ square pieces of equal size. Kalle and Juku play the following game. Each player at his move eats two pieces having a common side. Moves are made by turns, Juku starts. A player who cannot move loses. Who wins the game if

a) $m = 3$, $n = 3$;

b) $m = 2004$, $n = 2004$;
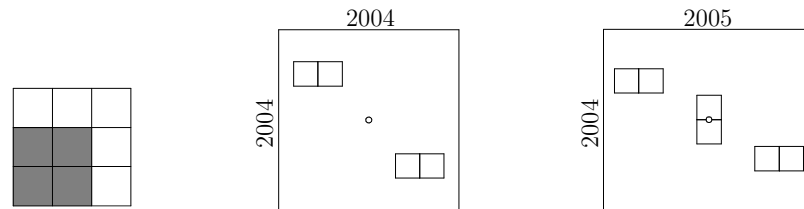
c) $m = 2004$, $n = 2005$?   *(Seniors.)*

Figure 8

2004

Figure 9

2005

Figure 10

Answer: a) Kalle; b) Kalle; c) Juku.

Solution. a) To Juku's first move, Kalle can reply with a move after which a part of shape 2×2 has been eaten up (see Figure 8). Irrespectively of Juku's second move, Kalle can make one more move. After that, Juku has no move.

b) When Juku eats up some two pieces, Kalle replies by eating two pieces which lie symmetrically with respect to the midpoint of the cake (see Figure 9). This guarantees that, after every move by Kalle, pieces not yet eaten up are situated symmetrically to the midpoint of the cake and, as it is not possible to eat two symmetric pieces at one move by the same player, Kalle can always follow the strategy described. Hence Juku's moves come first to the end.

c) On his first move, Juku can eat two pieces between which the midpoint of the cake lies and later use the strategy of Kalle from part b) (see Figure 10).

Remark. This game is called *Cram*, its full analysis for the case where length and width are both odd numbers seems to be quite complicated and is not completed yet.

10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + f(y)) = x + f(f(y))$$

for all real numbers x and y whereby $f(2004) = 2005$. *(Seniors.)*

Answer: $f(x) = x + 1$ is the only such function.

Solution 1. Taking $y = 0$, we get the equality $f(x + f(0)) = x + f(f(0))$. Making the substitution $x + f(0) = z$, we obtain $f(z) = z - f(0) + f(f(0))$ for every real number z. Hence f is a linear function $f(x) = x + a$. To find a, take $x = 2004$ in the last expression and, by using the known value of the function, obtain $a = 1$. A quick checking shows that the function $f(x) = x + 1$ satisfies the conditions of the problem.

Solution 2. Taking $x = -f(f(y))$, we see that $f(y) = 0$ for some y. Then $f(x) = x + f(0)$. From the condition $f(2004) = 2005$, we find $f(0) = 1$. Thus $f(x) = x + 1$.

11. Three rays are going out from point O in space, forming pairwise angles α, β and γ with $0° < \alpha \leqslant \beta \leqslant \gamma \leqslant 180°$. Prove that

$$\sin \frac{\alpha}{2} + \sin \frac{\beta}{2} > \sin \frac{\gamma}{2}.$$

(Seniors.)

5

6

Figure 11

Solution 1. Choose points A, B, and C on the three rays, respectively, so that $|OA| = |OB| = |OC| = d$ and $\angle BOC = \alpha$, $\angle COA = \beta$, $\angle AOB = \gamma$. These three points must be different and do not lie on the same line. From isosceles triangle BOC with vertex angle α and side length d (see Figure 11), we obtain $|BC| = 2d\sin\dfrac{\alpha}{2}$. Analogously from triangles COA and AOB, find $|CA| = 2d\sin\dfrac{\beta}{2}$ and $|AB| = 2d\sin\dfrac{\gamma}{2}$. As
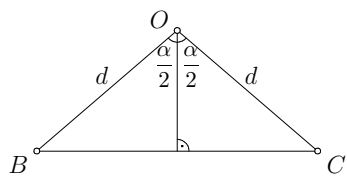
$$|BC| + |CA| > |AB|,$$

we obtain

$$2d\sin\frac{\alpha}{2} + 2d\sin\frac{\beta}{2} > 2d\sin\frac{\gamma}{2},$$

giving the desired inequality.

Solution 2. At first, we show that $\gamma \leqslant \alpha + \beta$. Consider the two of the given three rays which form angle of size γ, and build two cones by moving the third ray around both rays. On the plane defined by the axes, the first cone cuts angle α off from angle γ and the second cone cuts angle β. Assume $\gamma > \alpha + \beta$, then the last two angles do not overlap, therefore the cones have no common points except the vertex O, a contradiction. Thus $\gamma \leqslant \alpha + \beta$. On the other hand, $\alpha + \beta + \gamma \leqslant 360°$, giving

$$\frac{\gamma}{2} \leqslant \frac{\alpha}{2} + \frac{\beta}{2} \leqslant 180° - \frac{\gamma}{2}.$$

Therefore

$$\sin\frac{\gamma}{2} \leqslant \sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2} + \sin\frac{\beta}{2}$$

because $\dfrac{\alpha}{2}$, $\dfrac{\beta}{2}$ and $\dfrac{\gamma}{2}$ are grater than $0°$ but do not exceed $90°$, and also $\cos\dfrac{\beta}{2} < 1$ and $\cos\dfrac{\alpha}{2} < 1$.

12. We call a number triangle *amazing* if all its elements are different positive integers and, under every two neighbouring numbers, the quotient by division of the greater of two by the smaller is written. In the figure, one amazing triangle with side length 3 is shown. Find the smallest number which can occur as the greatest element in an amazing triangle with side length 4. *(Seniors.)*

21		84		7
	4		12	
		3		
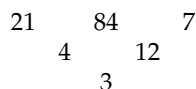
10		120		5		15
	12		24		3	
		2		8		
			4			

Figure 12

Solution. Let a_1, a_2, a_3, a_4 be the smallest numbers and b_1, b_2, b_3, b_4 be the biggest numbers of the first, second, third and fourth rows, respectively. Obviously $b_4 = a_4$ and $b_3 = a_3 b_4$. In the second row, there exists a number which is the product of the biggest element in the third row and some other element in the second. Thus $b_2 \geqslant a_2 b_3 = a_2 a_3 a_4$. Finally for the first row, we obtain similarily $b_1 \geqslant a_1 b_2 \geqslant a_1 a_2 a_3 a_4$. All numbers in the triangle are greater than 1, otherwise we could find two equal numbers in it. Since all numbers are different, we have $a_1 a_2 a_3 a_4 \geqslant 2 \cdot 3 \cdot 4 \cdot 5 = 120$. Hence $b_1 \geqslant 120$. Number 120 is achievable as follows from Figure 12.

Selected Problems from the Final Round of National Olympiad

1. Rein solved a test on mathematics that consisted of questions on algebra, geometry and logic. After checking the results, it occurred that Rein had answered correctly 50% of questions on algebra, 70% of questions on geometry and 80% of questions on logic. Thereby, Rein had answered correctly altogether 62% of questions on algebra and logic, and altogether 74% of questions on geometry and logic. What was the percentage of correctly answered questions throughout all the test by Rein? *(Grade 9.)*

Answer: 65%.

Solution: Let a, g, and l be the numbers of correctly answered questions on algebra, geometry and logic, and A, G, and L be the total number of questions on algebra, geometry and logic, respectively. The conditions of the problem imply $a = 0.5A$, $g = 0.7G$, $l = 0.8L$, $a + l = 0.62(A + L)$, $g + l = 0.74(G + L)$. After substituting to the fourth and fifth equation, we obtain $0.5A + 0.8L = 0.62A + 0.62L$, or equivalently $0.12A = 0.18L$, giving

$$A = 1.5L,$$

and $0.7G + 0.8L = 0.74G + 0.74L$, or equivalently $0.04G = 0.06L$, giving

$$G = 1.5L.$$

Now

$$a + g + l = 0.5A + 0.7G + 0.8L = 0.75L + 1.05L + 0.8L = 2.6L$$

and

$$A + G + L = 1.5L + 1.5L + L = 4L.$$

Thus the percentage of correct answers was

$$\frac{a + g + l}{A + G + L} = \frac{2.6}{4} = 65\%.$$

2. Represent the number

$$\sqrt[3]{1342\sqrt{167} + 2005}$$

in the form where it contains only addition, subtraction, multiplication, division and square roots. *(Grade 10.)*

Answer: $2\sqrt{167} + 1$.

Solution 1. First, isolate the terms divisible by 167:

$$1342\sqrt{167} + 2005 = 1336\sqrt{167} + 2004 + 6\sqrt{167} + 1 =$$
$$= 8 \cdot 167\sqrt{167} + 12 \cdot 167 + 6\sqrt{167} + 1.$$

Second, represent the result in the form

$$1342\sqrt{167} + 2005 = (2\sqrt{167})^3 + 3 \cdot (2\sqrt{167})^2 + 3 \cdot (2\sqrt{167}) + 1 =$$
$$= (2\sqrt{167} + 1)^3.$$

Therefore

$$\sqrt[3]{1342\sqrt{167} + 2005} = 2\sqrt{167} + 1.$$

Solution 2. Search the answer in the form $a\sqrt{167} + b$, where a and b are integers. Then we must have

$$(a\sqrt{167} + b)^3 = 1342\sqrt{167} + 2005,$$

or equivalently

$$167a^3\sqrt{167} + 3 \cdot 167a^2b + 3ab^2\sqrt{167} + b^3 = 1342\sqrt{167} + 2005.$$

Thus a and b must satisfy the system

$$\begin{cases} 167a^3 + 3ab^2 = 1342 \\ 501a^2b + b^3 = 2005 \end{cases}.$$

The second equation can be rewritten in the form $(501a^2 + b^2)b = 2005$. Since a and b differ from 0 and are integers, $501a^2 + b^2$ must be a divisor of 2005 that is greater than 501. The only possibility is now $501a^2 + b^2 = 2005$, giving $b = 1$, $a = 2$. Simple check shows that $a = 2$, $b = 1$ satisfy the first equation as well. Hence $(2\sqrt{167}+1)^3 = 1342\sqrt{167}+2005$.

3. A 5×5 board is covered by eight hooks (a three unit square figure, shown in the picture) so that one unit square remains free. Determine all squares of the board that can remain free after such covering. *(Grade 10.)*

Answer: All the squares that are colored dark in the Figure 13.
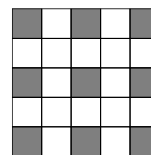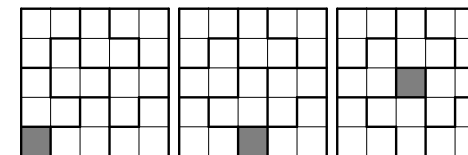

Figure 13   Figure 14

Solution. Suppose all dark squares are covered. Since one hook cannot cover more than one dark square, in total at least nine hooks are needed. As only eight of these are available, one of the dark squares must remain uncovered. There are three fundamentally different possibilities: the free square lies in the corner, in the middle of a side or in the centre of the board. The corresponding tilings are shown in the Figure 14.

4. Real numbers x and y satisfy the system of equalities

$$\begin{cases} \sin x + \cos y = 1 \\ \cos x + \sin y = -1 \end{cases}.$$

Prove that $\cos 2x = \cos 2y$. *(Grade 11.)*

Solution 1. After squaring both sides, we obtain

$$\sin^2 x + 2\sin x \cos y + \cos^2 y = 1,$$
$$\cos^2 x + 2\cos x \sin y + \sin^2 y = 1.$$

After adding the equations and dividing by 2, we obtain

$$\sin x \cos y + \sin y \cos x = 0,$$

or equivalently

$$\sin(x + y) = 0.$$

Hence $x + y = k\pi$, where k is integral. Therefore $2x = 2k\pi - 2y$, giving $\cos 2x = \cos 2y$.

Solution 2. After adding the equations, we obtain

$$\sin x + \sin y + \cos x + \cos y = 0,$$

which is equivalent to

$$2\sin\frac{x+y}{2}\cos\frac{x-y}{2} + 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} = 0,$$

implying

$$\cos\frac{x-y}{2}\left(\sin\frac{x+y}{2} + \cos\frac{x+y}{2}\right) = 0.$$

If $\cos \dfrac{x-y}{2} = 0$, then $\dfrac{x-y}{2} = (2k-1)\cdot\dfrac{\pi}{2}$, giving $2x - 2y = (2k-1)\cdot 2\pi$. Hence $\cos 2x = \cos 2y$. If $\sin\dfrac{x+y}{2} + \cos\dfrac{x+y}{2} = 0$, then $\tan\dfrac{x+y}{2} = -1$, implying $\dfrac{x+y}{2} = k\pi - \dfrac{\pi}{4}$ and $x + y = 2k\pi - \dfrac{\pi}{2}$. Therefore $\cos y = \cos\left(-\dfrac{\pi}{2} - x\right) = \cos\left(\dfrac{\pi}{2} + x\right) = -\sin x$, leading to $\sin x + \cos y = 0$ that contradicts the first equation of the initial system.

5. Let a, b, and n be integers such that $a + b$ is divisible by n and $a^2 + b^2$ is divisible by n^2. Prove that $a^m + b^m$ is divisible by n^m for all positive integers m. *(Grade 11.)*

Solution 1. We prove that a and b are divisible by n, then the claim immediately follows. As

$$2ab = (a+b)^2 - (a^2 + b^2),$$

$2ab$ is divisible by n^2. Let p be any prime in the prime decomposition of n and let α be its exponent. Then the exponent of p is at least 2α in the prime decomposition of $2ab$, and at least $2\alpha - 1$ in the prime decomposition of ab. Therefore at least one of numbers a and b must be divisible by p^α. As $a+b$ is divisible by n and hence by p^α, also the other of the numbers a and b must be divisible by p^α. Altogether, this means that both a and b are divisible by n.

6. A post service of some country uses carriers to transport the mail; each carrier's task is to bring the mail from one city to a neighbouring city. It is known that it is possible to send mail from any city to the capital P. For any two cities A and B, call B *more important than* A, if every possible route of mail from A to the capital P goes through B.

a) Prove that, for any three different cities A, B, and C, if B is more important than A and C is more important than B, then C is more important than A.

b) Prove that, for any three different cities A, B, and C, if both B and C are more important than A, then either C is more important than B or B is more important than C. *(Grade 11.)*

Solution. a) Let t be any possible mail route from A to P. Since B is more important than A, the route t goes through B. The end part of t from B to P is a mail route from B to P. Since C is more important than B, this route goes through C. Therefore, t goes through C.

b) Assume that the claim doesn't hold, that is, C is not more important than B and B is not more important than C. Then there exist a route from B to P not going through C and a route from C to P not going through B. Consider any route from A to P. Since B and C are more important than A, this route goes through both B and C. Start moving from A along this route and find out which of the cities B and C comes up first. If it is B, then continue along the route to P that doesn't pass through C. So we have found a route from A to P that doesn't go through C, a contradiction with the

assumption that C is more important than A. Analogously, if C comes up first, then we get a contradiction with the assumption that B is more important than A. Thus our original assumption was false.

Remark. In graph theory, the relation "is more important than" of this problem is called *postdominance*.

7. In a fixed plane, consider a convex quadrilateral $ABCD$. Choose a point O in the plane and let K, L, M, and N be the circumcentres of triangles AOB, BOC, COD, and DOA, respectively. Prove that there exists exactly one point O in the plane such that $KLMN$ is a parallelogram. *(Grade 11.)*

Solution. If O is the point described in the problem, then we must have $KL \perp BO$ because K and L both lie on the perpendicular bisector of BO. Similarly $LM \perp CO$, $MN \perp DO$, and $NK \perp AO$. Let O be the intersection point of the diagonals of $ABCD$ (see Figure 15). Then both KL and MN are perpendicular to BD, giving $KL \parallel MN$. Similarly $LM \parallel NK$. Therefore the opposite sides of $KLMN$ are parallel, meaning that $KLMN$ is a parallelogram. On the other hand, if O is a point for which $KLMN$ is a parallelogram, we have $KL \parallel MN$. Then also $BO \parallel DO$, giving that O lies on the line BD. We can show similarily that O lies also on the line AC. Therefore O is the intersection point of the diagonals.
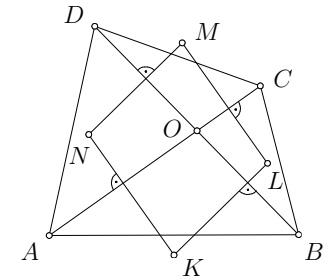

Figure 15

8. Does there exist an integer $n > 1$ such that

$$2^{2^n-1} - 7$$

is not a perfect square? *(Grade 11.)*

Answer: Yes.

Solution 1. Let us show that if $n = 5$, then the number $2^{2^n-1} - 7$ is not a perfect square. Note that $2^{10} = 1024 \equiv 1 \pmod{11}$, giving $2^{31} = 2\cdot(2^{10})^3 \equiv 2 \pmod{11}$. Hence the remainder of division of $2^{31} - 7$ by 11 is 6. On the other hand, squares of integers can have remainders 0, 1, 4, 9, 5, and 3 in division by 11.

Solution 2. Computation gives $2^{31} - 7 = 32768\cdot 65536 - 7 = 2147483641$ but $46340^2 = 2147395600$ and $46341^2 = 46340^2 + 2\cdot 46340 + 1 = 2147488281$. So $46340^2 < 2^{31} - 7 < 46341^2$.

Remark. The number 11 is the least modulus with respect to which $2^{2^5-1} - 7$ is not a quadratic residue. There exist greater such numbers, e.g. 31.

9. Punches in the buses of a certain bus company always always cut exactly six holes into the ticket. The possible locations of the holes form a 3×3 table as shown in the figure. Mr. Freerider wants to put together a collection of tickets such that, for any combination of punch holes, he would have a ticket with the same combination in his collection. The ticket can be viewed both from the front and from the back. Find the smallest number of tickets in such a collection. *(Grade 12.)*
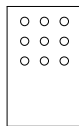
Answer: 47.

Solution. Instead of holes, we can deal with non-holes — the locations that are not cut through during punching. The number of possibilities to choose 3 locations for non-holes from 9 locations is

$$\binom{9}{3} = 84.$$

One ticket can represent either one punch combination that is symmetric with respect to the central axis parallel to the longer sides of the ticket or two different combinations that are mirror images of each other with respect to this axis. In the case of symmetric combinations, either all three non-holes lie in the second column (1 possibility) or one non-hole lies in the second column and other two lie in the same rows, one in the first column and the other in the second ($3 \cdot 3 = 9$ possibilities). So there are $1 + 9 = 10$ symmetric combinations and $84 - 10 = 74$ non-symmetric combinations. The number of tickets needed to cover these combinations is

$$10 + \frac{74}{2} = 47.$$

10. Consider a convex n-gon in the plane with n being odd. Prove that if one may find a point in the plane from which all the sides of the n-gon are viewed at equal angles, then this point is unique. (We say that segment AB is *viewed at angle γ from point O* iff $\angle AOB = \gamma$.) *(Grade 12.)*

Solution. Draw the rays from the point described in the problem through the vertices of the polygon. The point can lie either inside or outside the polygon, therefore there are two possibilities for the rays: they divide either all the plane or only an angle into equal angles (see Figure 16). The latter case would imply that the outermost rays were both incident to one vertex and all the others were incident to two vertices of the polygon, giving that n is even. This contradiction shows that the point satisfying the conditions of the problem lies inside the polygon.

Assume now that there are two different points P and Q inside $A_1 A_2 \ldots A_n$, from which all the sides are viewed at equal angles. Then for every $i = 1, 2, \ldots, n$ (taking $A_{n+1} = A_1$),

$$\angle A_i P A_{i+1} = \frac{2\pi}{n}, \qquad \angle A_i Q A_{i+1} = \frac{2\pi}{n}.$$
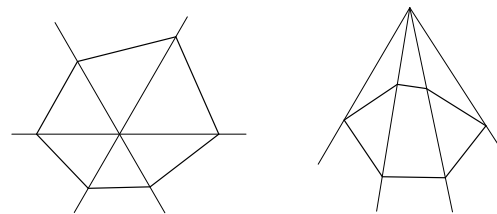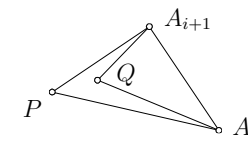
Figure 16

Figure 17

As P and Q are different, there exist such vertices of the polygon A_i and A_{i+1} that the point Q is inside or on the side of $A_i P A_{i+1}$, not coinciding with the vertex P (see Figure 17). But this implies $\angle A_i Q A_{i+1} > \angle A_i P A_{i+1}$, a contradiction. Hence there exists only one point satisfying the conditions of the problem.

Remark. For $n = 4, 6$ one may find several different points satisfying the conditions of the problem.

11. A string having a small loop in one end is set over a horizontal pipe so that the ends hang loosely. After that, the other end is put through the loop, pulled as far as possible from the pipe and fixed in that position whereby this end of the string is farther from the pipe than the loop. Let α be the angle by which the string turns at the point where it passes through the loop (see picture). Find α. *(Grade 12.)*
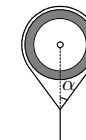
Answer: $\dfrac{\pi}{3}$.

Solution. Let O and r be the centre point and the radius of the pipe, respectively. Let l be the length of the string, A and B the loose end and the end with the loop, respectively, and let C and D be the first and the last tangent point with the surface of the pipe (see Figure 18). At first, find the length of AO, denote by $d(\alpha)$. Obviously $|BC| = |BD| = r \cot \alpha$. As $\angle COD = \pi - 2\alpha$, the string touches the pipe along the arc CD at angle $\pi + 2\alpha$ and thus the length of the string along the arc CD is $r(\pi + 2\alpha)$. After subtracting the lengths of segments BC and BD and the arc CD from the total length of the string, we obtain

$$|AB| = l - 2r \cot \alpha - r(\pi + 2\alpha).$$

We also have

$$|BO| = \frac{r}{\sin \alpha}.$$

Altogether $d(\alpha) = |AB| + |BO|$, or equivalently

$$d(\alpha) = l - 2r \cot \alpha - r(\pi + 2\alpha) + \frac{r}{\sin \alpha}.$$

Now we find α for which the value of $d(\alpha)$ is the greatest. The derivative equals

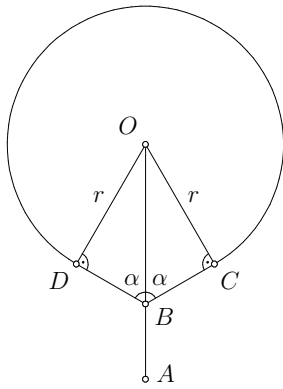$$d'(\alpha) = \frac{2r}{\sin^2 \alpha} - 2r - \frac{r \cos \alpha}{\sin^2 \alpha},$$

Figure 18

or equivalently,

$$d'(\alpha) = r\left(\frac{2 - 2\sin^2\alpha - \cos\alpha}{\sin^2\alpha}\right) = r\left(\frac{2\cos^2\alpha - \cos\alpha}{\sin^2\alpha}\right).$$

The condition $d'(\alpha) = 0$ gives the equation $2\cos^2\alpha - \cos\alpha = 0$, implying $\cos\alpha = 0$ or $\cos\alpha = \frac{1}{2}$. Thus $\alpha = \frac{\pi}{2}$ or $\alpha = \frac{\pi}{3}$, bearing in mind that $0 < \alpha \leqslant \frac{\pi}{2}$. In order to find the maximal value, we consider $d'(\alpha)$ in the neighbourhoods of α found out before. The denominator $\sin^2\alpha$ is positive in these neighbourhoods, the numerator $2\cos^2\alpha - \cos\alpha$ is negative iff $0 < \cos\alpha < \frac{1}{2}$. Therefore $d'(\alpha)$ is positive and $d(\alpha)$ is increasing in the interval $0 < \alpha < \frac{\pi}{3}$, $d'(\alpha)$ is negative and $d(\alpha)$ is decreasing in the interval $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$. Hence the function $d(\alpha)$ obtains its maximal value at $\alpha = \frac{\pi}{3}$.

Remark. The value for α found out in the solution is attainable iff the ratio between the length of the string and the diameter of the pipe is at least $\frac{5\pi + 2\sqrt{3}}{6}$.

12. A sequence of natural numbers a_1, a_2, a_3, ... is called *periodic modulo n* if there exists a positive integer k such that, for any positive integer i, the terms a_i and a_{i+k} are equal modulo n. Does there exist a strictly increasing sequence of natural numbers that

a) is not periodic modulo finitely many positive integers and is periodic modulo all the other positive integers;

b) is not periodic modulo infinitely many positive integers and is periodic modulo infinitely many positive integers? *(Grade 12.)*

Answer: a) no; b) yes.

Solution. a) Suppose that the sequence a_1, a_2, a_3, ... is not periodic modulo finitely many positive integers, let N be the largest of these. Since $2N > N$, this sequence must

be periodic modulo $2N$. On the other hand, if a_i and a_j are congruent modulo $2N$, they are also congruent modulo N. Therefore, the sequence is periodic modulo N, a contradiction.

b) We are going to prove that the sequence $a_i = 2^{i-1}$ is periodic modulo no even numbers and all odd numbers. Let first the modulus N be even. If the sequence were periodic modulo N, then, using a similar argument as in a), we get that the sequence would be periodic modulo any factor of N. But the sequence $1, 0, 0, 0, \ldots$ of remainders modulo 2 is not periodic, so our sequence cannot be periodic modulo N. Now let the modulus N be odd. As the set of possible remainders in division by N is finite, there exist two indices i and j with $i < j$ such that $a_i = 2^{i-1}$ and $a_j = 2^{j-1}$ are congruent modulo N. Then the difference $2^{j-1} - 2^{i-1} = 2^{i-1}(2^{j-i} - 1)$ is divisible by N. Since N is odd, $2^{j-i} - 1$ must be divisible by N. Hence $a_1 = 1$ and $a_{j-i+1} = 2^{j-i}$ are congruent modulo N, implying also that $a_2 = 2a_1$ and $a_{j-i+2} = 2a_{j-i+1}$ are congruent modulo N, that $a_3 = 2a_2$ and $a_{j-i+3} = 2a_{j-i+2}$ are congruent modulo N etc, that is, the sequence is periodic modulo N.

13. A *crymble* is a solid consisting of four white and one black unit cubes as shown in the picture. Find the side length of the smallest cube that can be exactly filled up with crymbles. *(Grade 12.)*

Answer: 10.

Solution. Since a crymble consists of 5 unit cubes, the volume of the cube made up from crymbles and hence also the length of its side must be divisible by 5. A cube with the side length 5 cannot be filled up with crymbles. To prove this, colour 27 unit cubes as shown in the Figure 19. One crymble cannot fill more than one coloured cube, therefore at least 27 crymbles are needed. But their volume $27 \cdot 5 = 135$ is larger than the volume $5^3 = 125$ of the cube.

A cube with side length 10 can be filled up with crymbles. By putting together two crymbles, construct the solid that is in the Figure 20. From two such solids, make a $2 \times 2 \times 5$ cuboid. From such cuboids, it is possible to put together a $10 \times 10 \times 10$ cube.
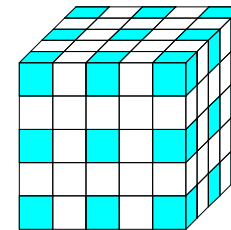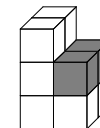

Figure 19

Figure 20

IMO Team Selection Contest

First Day

1. On a plane, a line l and two circles c_1 and c_2 of different radii are given such that l touches both circles at point P. Point $M \neq P$ on l is chosen so that the angle $Q_1 M Q_2$ is as large as possible where Q_1 and Q_2 are the tangency points of the tangent lines drawn from M to c_1 and c_2, respectively, differing from l. Find $\angle PMQ_1 + \angle PMQ_2$.

Answer: π.

Solution. Consider first the case where c_1 and c_2 are on the same side from l (see Figure 21). Let O_1 and O_2 be the circumcentres and r_1 and r_2 the radii of c_1 and c_2, respectively. Without loss of generality, assume $r_1 > r_2$. Denote $\angle PMQ_1 = \alpha_1$, $\angle PMQ_2 = \alpha_2$ and $|PM| = d$. As $\angle PMO_1 = \frac{1}{2}\angle PMQ_1$ and $\angle PMO_2 = \frac{1}{2}\angle PMQ_2$, we see that $\angle Q_1 M Q_2 = \alpha_1 - \alpha_2$ is maximal if and only if $\angle O_1 M O_2 = \frac{\alpha_1}{2} - \frac{\alpha_2}{2}$ is maximal; the latter holds if and only if $\tan\left(\frac{\alpha_1}{2} - \frac{\alpha_2}{2}\right)$ is maximal because the angle is in the first quadrant. The formula of tangent of difference gives

$$\tan\left(\frac{\alpha_1}{2} - \frac{\alpha_2}{2}\right) = \frac{\tan\frac{\alpha_1}{2} - \tan\frac{\alpha_2}{2}}{1 + \tan\frac{\alpha_1}{2} \cdot \tan\frac{\alpha_2}{2}} = \frac{\frac{r_1}{d} - \frac{r_2}{d}}{1 + \frac{r_1}{d} \cdot \frac{r_2}{d}}.$$

Representing the result in the form

$$\tan\left(\frac{\alpha_1}{2} - \frac{\alpha_2}{2}\right) = \frac{r_1 - r_2}{d + \frac{r_1 r_2}{d}} = \frac{r_1 - r_2}{\sqrt{r_1 r_2}\left(\frac{d}{\sqrt{r_1 r_2}} + \frac{\sqrt{r_1 r_2}}{d}\right)},$$

we obtain that the value of the denominator of the last expression is minimal in the case $d = \sqrt{r_1 r_2}$. Now

$$\tan\frac{\alpha_1}{2} = \frac{r_1}{d} = \frac{\sqrt{r_1}}{\sqrt{r_2}}, \qquad \tan\frac{\alpha_2}{2} = \frac{r_2}{d} = \frac{\sqrt{r_2}}{\sqrt{r_1}},$$

i.e. $\tan\frac{\alpha_1}{2}$ and $\tan\frac{\alpha_2}{2}$ are reciprocals of each other. Therefore $\frac{\alpha_1}{2} + \frac{\alpha_2}{2} = \frac{\pi}{2}$ and $\angle PMQ_1 + \angle PMQ_2 = \alpha_1 + \alpha_2 = \pi$.

In the other case when c_1 and c_2 are on the different sides from l (see Figure 22), the maximal size of the angle $Q_1 M Q_2$ is π which is the greatest size an angle can have. In this case, the point M lies on the common tangent to c_1 and c_2 intersecting l. Then $\angle PMQ_1 + \angle PMQ_2 = \angle Q_1 M Q_2 = \pi$.

2. On the planet Automory, there are infinitely many inhabitants. Every Automorian loves exactly one Automorian and honours exactly one Automorian. Additionally, the following can be noticed:
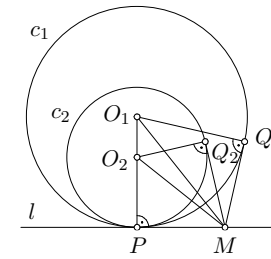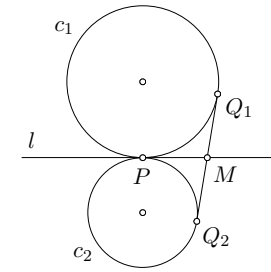

Figure 21　　　　　　　　Figure 22

- each Automorian is loved by some Automorian;
- if Automorian A loves Automorian B, then also all Automorians honouring A love B;
- if Automorian A honours Automorian B, then also all Automorians loving A honour B.

Is it correct to claim that every Automorian honours and loves the same Automorian?

Answer: Yes.

Solution. Denote by $f(A)$ the Automorian loved by A and by $g(A)$ the Automorian honoured by A. The conditions of the problem imply the following:

- for every Automorian A, there exists an Automorian C such that $f(C) = A$;
- for every Automorian C, $f(g(C)) = f(C)$;
- for every Automorian C, $g(f(C)) = g(C)$.

We will show that $f(A) = g(A)$ for every A. Applying f to both sides of the third condition, we get

$$f(g(f(C))) = f(g(C)).$$

Using the second condition in both sides, we get

$$f(f(C)) = f(C).$$

Using the first condition, this implies

$$f(A) = A$$

for all A. Using the second condition again, we obtain the desired result:

$$f(A) = f(g(A)) = g(A).$$

Thus, every Automorian loves and honours himself.

3. Find all pairs (x, y) of positive integers satisfying the equation

$$(x + y)^x = x^y.$$

Answer: $(2,6)$ and $(3,6)$.

Solution. We have $x^y = (x+y)^x > x^x$ implying $y > x$. Let $y = nx$ where $n > 1$ is a rational number. From the equality given in the problem, we get

$$(x + nx)^x = x^{nx}.$$

Raise both sides to the power of $\dfrac{1}{x}$ and then divide them by x; we obtain

$$1 + n = x^{n-1}. \tag{1}$$

On the right hand side, the exponent $n - 1$ can be represented as a reduced fraction $\dfrac{p}{q}$,

therefore the number

$$x^{n-1} = x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

is either natural or irrational. The left hand side of (1) cannot be irrational, thus it is natural. Hence n is natural. By choice, $n > 1$.

If $n = 2$, (1) gives $x = 3$ implying $y = 2x = 6$. If $n = 3$, (1) gives $x = 2$ implying $y = 3x = 6$. Note that (1) implies $x \geqslant 2$, thus $1 + n \geqslant 2^{n-1}$. The latter inequality does not hold in the case $n \geqslant 4$. Hence no more solutions exist.

Second Day

4. Find all pairs (a, b) of real numbers such that the roots of polynomials $6x^2 - 24x - 4a$ and $x^3 + ax^2 + bx - 8$ are all non-negative real numbers.

Answer: $(-6, 12)$.

Solution. Let x_1, x_2 be the roots of the first polynomial and x_1', x_2', x_3' be the roots of the other polynomial. Division of the first polynomial by 6 gives $x^2 - 4x - \dfrac{2}{3}a$ whose roots are x_1 and x_2, too. By Viète's formulae,

$$x_1 + x_2 = 4, \qquad x_1 x_2 = -\frac{2}{3}a$$

and

$$x_1' + x_2' + x_3' = -a, \qquad x_1' x_2' + x_2' x_3' + x_3' x_1' = b, \qquad x_1' x_2' x_3' = 8.$$

Now

$$4 = \left(\frac{4}{2}\right)^2 = \left(\frac{x_1 + x_2}{2}\right)^2 \geqslant x_1 x_2 = -\frac{2}{3}a$$

and

$$-\frac{2}{3}a = \frac{2}{3}(x_1' + x_2' + x_3') \geqslant 2\sqrt[3]{x_1' x_2' x_3'} = 4.$$

We see that in both inequalities, equality actually holds. Consequently, $x_1 = x_2$, $x_1' = x_2' = x_3'$, and $-\dfrac{2}{3}a = 4$. From the latter, we find $a = -6$. Thus $x_1' x_2' x_3' = 8$ leading to $x_1' = x_2' = x_3' = 2$ which gives $b = 12$.

On the other hand, taking $a = -6$, $b = 12$ gives $6x^2 - 24x - 4a = 6(x - 2)^2$ and $x^3 + ax^2 + bx - 8 = (x - 2)^3$ satisfying the conditions of the problem.

5. On a horizontal line, 2005 points are marked, each of which is either white or black. For every point, one finds the sum of the number of white points on the right of it and the number of black points on the left of it. Among the 2005 sums, exactly one number occurs an odd number of times. Find all possible values of this number.

Answer: 1002.

Solution. It is easy to see that the sums computed for a white point V and a black point M immediately following V on its right are equal. Note also that the sums are equal also if the two points of different colour lie in the opposite order. If one interchanges two consecutive points of different colour, only the two equal sums corresponding to these two points change giving rise to two new equal sums. Hence, for any k, such transitions preserve parity of the number of occurrences of k among the 2005 sums.

Assume there are n white and $2005 - n$ black points on the line. By a sequence of transitions described, collect all white points to the left. Then, going from left to right, the corresponding sums are

$$n - 1, n - 2, \ldots, 1, 0, 0, 1, \ldots, 2003 - n, 2004 - n.$$

According to the invariant discovered in the first paragraph, exactly one number must occur an odd number of times also in this sequence. As the middle numbers occur in pairs, the single number occurring an odd number of times is either the leftmost $n - 1$ or the rightmost $2004 - n$. If the former case takes place, then $n - 2 = 2004 - n$ leading to $n = 1003$ and $n - 1 = 1002$. If the other case takes place, then $n - 1 = 2003 - n$, giving $n = 1002$ and $2004 - n = 1002$ just like in the first case.

6. On a plane, line l and a circle having no common points are given. Let AB be the diameter of the circle being perpendicular to l whereby B is nearer to l than A. Let C be a point on the circle different from both A and B. Line AC intersects l at point D. Points B and E, the latter obtained as the tangency point of a line drawn from D to the circle, lie on the same side from AC. Line EB intersects l at point F; line FA intersects the circle second time at point G. Prove that the point symmetric to G with respect to AB lies on FC.

Solution. See IMO-2004 Shortlist.