Estonian math competitions 2002/2003

We thank the IMO community for many of these problems which have been taken from various materials distributed at the recent IMO-s.

Autumn Open Contest: October 2002

Juniors (up to 10th grade)

1. All inner angles of a 7-gon are obtuse, their sizes in degrees being pairwise different integer numbers divisible by 9. Find the sum of the two biggest angles of this 7-gon.

2. Circles with centres O_1 and O_2 intersect in two points, let one of which be A. The common tangent of these circles touches them respectively in points P and Q. It is known that points O_1, A and Q are on a common straight line and points O_2, A and P are on a common straight line. Prove that the radii of the circles are equal.

3. A 4-digit number $\overline{abcd}$ is divisible by the product of 2-digit numbers $\overline{ab}$ and $\overline{cd}$. Find all 4-digit numbers with such property.

4. Mari and Jüri ordered a round pizza. Jüri cut the pizza into four pieces by two straight cuts, none of which passed through the centre point of the pizza. Mari can choose two pieces not aside of these four, and Jüri gets the rest two pieces. Prove that if Mari chooses the piece that covers the centre point of the pizza, she will get more pizza than Jüri.

5. The clock on the wall shows four numbers (possible times from 00:00 to 23:59), the shapes of the numbers being as shown in the picture.
   a) How many times a day does the mirror image of the clock on the wall correspond to some time (the mirror image of number 1 is equal to number 1)?
   b) How many times a day does the mirror image of the clock on the wall correspond to the same time as the clock?

Seniors (grades 11 and 12)

1. Find all quadruples of integer numbers (a, b, c, d) for which
   $$52^a \cdot 77^b \cdot 88^c \cdot 91^d = 2002 .$$

2. Four rays spread out from point O in a 3-dimensional space in a way that the angle between every two rays is α. Find $\cos \alpha$.

3. Three consecutive positive integers each happen to be a power of some prime number. Find all triples of integers with this property.

4. n points of integral coordinates in the plane have been painted white. If some points A and B are white, we may paint white the point symmetrical to A in respect to B (at every step we paint only one new point). For which smallest natural number n can the initial n points be chosen in a way that any point of integral coordinates could be painted white in a finite number of steps?

5. During an election campaign K tabloid newspapers published compromising material about P politicians; every politician being written about in an odd number of newspapers and every newspaper writing about an odd number of politicians.
   a) Prove that the number of newspapers K and the number of politicians P are either both odd or both even numbers.
   b) Find the total number of all possibilities which newspaper wrote about which politicians.

Solutions of Autumn Open Contest

J1. *Answer:* 315°.

Let the interior angles of the 7-gon in degrees be $9a_1, 9a_2, \ldots, 9a_7$, where $a_1, a_2, \ldots, a_7$ are integers and $a_1 < a_2 < \ldots < a_7$. As all the interior angles are obtuse, we have $a_1 \geqslant 11$ and
$$a_1 + a_2 + \ldots + a_7 \geqslant 11 + 12 + 13 + 14 + 15 + 16 + 17 = 98 .$$

On the other hand $a_1 + a_2 + \ldots + a_7 = 100$, because the sum of the interior angles of a 7-gon is $(7 - 2) \cdot 180° = 900°$. Bearing in mind that $a_1 < a_2 < \ldots < a_7$ we find that the only possibilities are $(a_1, a_2, \ldots, a_7) = (11, 12, 13, 14, 15, 17, 18)$ and $(a_1, a_2, \ldots, a_7) = (11, 12, 13, 14, 15, 16, 19)$. In either case the sum of the two biggest angles is $9 \cdot (a_6 + a_7) = 9 \cdot 35 = 315°$.

J2. *Solution 1.* By construction $\angle O_1 PQ = \dfrac{\pi}{2} = \angle PQO_2$. We also have $|O_1P| = |O_1A|$ and $|O_2Q| = |O_2A|$ that implies
$$\angle O_1 P A = \angle P A O_1 = \angle O_2 A Q = \angle A Q O_2 .$$

Consequently $\angle AO_1P = \angle QO_2A$ and

$$\angle APQ = \frac{\pi}{2} - \angle O_1PA = \frac{\pi}{2} - \angle AQO_2 = \angle PQA \ .$$

Thus the triangles O_1PQ and O_2QP are similar because the respective angles are equal. As PQ is their common side, these triangles are congruent, giving $|O_1P| = |O_2Q|$.

Solution 2. Let $\angle O_1QP = \angle AQP = \alpha$. As $\angle O_1PQ = 90°$, we have $\angle PO_1A = \angle PO_1Q = 90° - \alpha$. Because the triangle PO_1A is isosceles, we obtain

$$\angle O_1PA = \angle O_1AP = \frac{1}{2}\bigl(180° - (90° - \alpha)\bigr) = 45° + \frac{\alpha}{2} \ .$$

On the other hand $\angle PQO_2 = 90°$ implies $\angle AQO_2 = 90° - \alpha$. The triangle AQO_2 is isosceles, consequently $\angle AQO_2 = 90° - \alpha$. The conditions of the problem imply $\angle O_2AQ = \angle O_1AP$, therefore

$$45° + \frac{\alpha}{2} = 90° - \alpha \ ,$$

giving $\alpha = 30°$. Thus $\angle PO_1A = 60°$ and $\angle O_1AP = \angle O_2AQ = 60°$, i.e. the triangles O_1AP and O_2AQ are equilateral. As $\angle O_1PQ = \angle PQO_2 = 90°$, the distance of the point A to the straight line PQ is equal to $\frac{1}{2}|O_1P|$. On the other hand the distance equals $\frac{1}{2}|O_2Q|$, giving us $|O_1P| = |O_2Q|$.
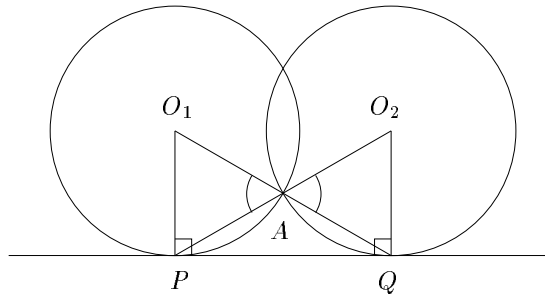

Figure 1

J3. *Answer:* 1352, 1734.

We need the integer $\overline{abcd} = 100 \cdot \overline{ab} + \overline{cd}$ be divisible by $\overline{ab}$, therefore $\overline{cd}$ must be divisible by $\overline{ab}$. Let $\overline{cd} = n \cdot \overline{ab}$, then n is a one-digit number (since $a \neq 0$). The necessary condition that $\overline{abcd} = (100 + n) \cdot \overline{ab}$ would be divisible by $\overline{ab} \cdot \overline{cd}$

is that $100 + n$ is divisible by $\overline{cd}$. We have obtained $100 + n = k \cdot \overline{cd} = k \cdot n \cdot \overline{ab}$ (where $k > 1$, because $100 + n$ is a 3-digit and $\overline{cd}$ is a 2-digit number). Therefore 100 must be divisible by a one-digit number n, implying n equals 1, 2, 4 or 5. Consider these four cases separately.

1) Case $n = 1$ gives the equation $101 = k \cdot \overline{ab}$. As 101 is prime and $k > 1$, this case has no solutions.

2) Case $n = 2$ gives $102 = k \cdot 2 \cdot \overline{ab}$ that implies $51 = k \cdot \overline{ab}$. As $51 = 3 \cdot 17$, the only possibility is $k = 3$, $\overline{ab} = 17$, and we obtain one 4-digit answer 1734.

3) Case $n = 4$ gives $104 = k \cdot 4 \cdot \overline{ab}$ that implies $26 = k \cdot \overline{ab}$. As $26 = 2 \cdot 13$, the only possibility is $k = 2$, $\overline{ab} = 13$, and we obtain one 4-digit answer 1352.

4) Case $n = 5$ gives $105 = k \cdot 5 \cdot \overline{ab}$ that implies $21 = k \cdot \overline{ab}$. As $21 = 3 \cdot 7$, this case has no solutions (the number 21 cannot be expressed as a product of a 2-digit number and a number exceeding 1).

J4. Construct a straight line parallel to one of the cuts passing through the centre point of the pizza O — this line passes through two pieces of all four, dividing both into two parts. Let a, b, c and d be areas as shown in the figure 2. If Mari chooses the piece having the centre point of the pizza, the total area of her share is $(a + b) + (c - d)$ and the total share of Jüri is $(a - b) + (c + d)$. To complete the solution, one has to prove that $a + b + c - d > a - b + c + d$, or $b - d > d - b$. This is similar to $b > d$ that evidently holds.
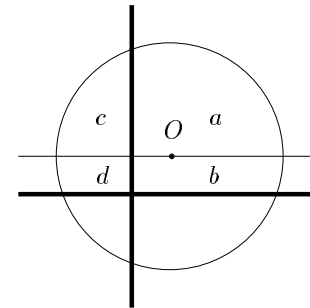

Figure 2

J5. *Answer:* a) 121; b) 11.

a) The numbers having a number as its mirror image are 0, 1, 2, 5 and 8. The number 8 can only be the last digit of hours or minutes, the mirror image of which is respectively the first digit of minuts or hours, implying that 8 cannot be in any time considered in the problem. Thus the first digit of hours can be 0, 1 or 2 and the last digit of hours can be 0, 1, 2 or 5, totally giving $3 \cdot 4 - 1 = 11$ numbers (25 is not an hour) as hour time. Since the mirror image of minutes is hours, the minutes can be any of these 11 numbers. Consequently the mirror image of the clock corresponds to some time $11 \cdot 11 = 121$ times a day.

3

4

b) For having the mirror image time equal to the real image time, the minutes must equal the mirror of the hours, i.e. any of the hours possible found in a) gives exactly one such time. Therefore the mirror image time equals the real image time 11 times a day.

S1. *Answer:* the only such quadruple is $(2, 2, -1, -1)$.

After factorisation of the bases of powers and 2002, we may rewrite the given equation as

$$(2^{2a} \cdot 13^a) \cdot (7^b \cdot 11^b) \cdot (2^{3c} \cdot 11^c) \cdot (7^d \cdot 13^d) = 2 \cdot 7 \cdot 11 \cdot 13$$

or in the form

$$2^{2a+3c} 7^{b+d} 11^{b+c} 13^{a+d} = 2 \cdot 7 \cdot 11 \cdot 13.$$

For the last equation to hold, the powers of respective primes on the left and right side must be equal. This implies a system of equations

$$\begin{cases} 2a + 3c = 1 \\ b + d = 1 \\ b + c = 1 \\ a + d = 1 \end{cases}.$$

The second and the third equation give $c = d$, and the second and the fourth $a = b$. Multiplying the third equation by 2 and subtracting from the first gives $c = -1$ and $b = 2$. The rest of the unknown variables must then be $a = 2$ and $d = -1$. We have obtained the only quadruple satisfying the conditions $(2, 2, -1, -1)$.

S2. *Answer:* $-\dfrac{1}{3}$.

Consider the points A_1, A_2, A_3 and A_4 on the respective rays equidistant from O. Consider any pair of points A_{i_1}, A_{j_1} and A_{i_2}, A_{j_2}, where $i_1 \neq j_1$ and $i_2 \neq j_2$. As $|OA_{i_1}| = |OA_{i_2}|$ and $|OA_{j_1}| = |OA_{j_2}|$, also $\angle A_{i_1} O A_{j_1} = \alpha = \angle A_{i_2} O A_{j_2}$, we obtain that the triangles $OA_{i_1} A_{j_1}$ and $OA_{i_2} A_{j_2}$ are congruent, implying $|A_{i_1} A_{j_1}| = |A_{i_2} A_{j_2}|$. Therefore all segments $A_i A_j$, where $i \neq j$ are of equal lengths. Consequently all the faces of $A_i A_j A_k$, having i, j, k pairwise different, are equilateral triangles, and $A_1 A_2 A_3 A_4$ is a regular tetrahedron.

Cut this tetrahedron with a plane ω containing the points O, A_1 and A_2. The straight line $A_1 O$ passes through the triangle $A_2 A_3 A_4$ in its centre M_1 and the straight line $A_2 O$ passes through the triangle $A_1 A_3 A_4$ in its centre M_2 (see figure 3). By construction M_1 and M_2 are on the plane ω. The straight line $A_2 M_1$ as a median of the triangle $A_2 A_3 A_4$ bisects the segment $A_3 A_4$ and the straight line $A_1 M_2$ as a median of the triangle $A_1 A_3 A_4$ bisects the segment $A_3 A_4$ as well.

Hence the centre point K of the segment $A_3 A_4$ is also on the plane ω, having $|KA_1| = |KA_2|$. As $A_1 M_1 \perp KA_2$ and $A_2 M_2 \perp KA_1$, we have that $KM_1 O M_2$ is a cyclic quadrangle that implies

$$\alpha = \angle A_1 O A_2 = \angle M_1 O M_2 = 180° - \angle M_1 K M_2.$$

As KA_2 is a median of $A_2 A_3 A_4$, the ratio $|KM_1| : |KA_2| = 1 : 3$, therefore $\cos \angle M_1 K M_2 = \dfrac{|KM_1|}{|KA_1|} = \dfrac{|KM_1|}{|KA_2|} = \dfrac{1}{3}$. Finally we obtain $\cos \alpha = -\cos \angle M_1 K M_2 = -\dfrac{1}{3}$.
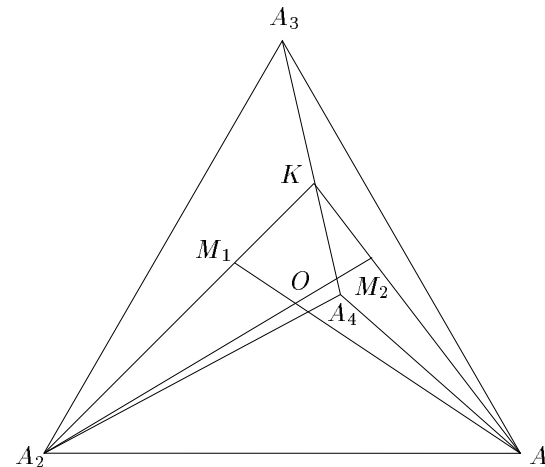
Figure 3

S3. *Answer:* $(1, 2, 3)$, $(2, 3, 4)$, $(3, 4, 5)$ and $(7, 8, 9)$.

At least one of the three consecutive integers is divisible by 2 and hence must be a power of 2 (with positive exponent). Considering $2^1 = 2$, $2^2 = 4$ and $2^3 = 8$, we find the solutions. We prove that no more solutions exist. Say 2^k belongs to a triple in question, where $k \geqslant 4$. Since the only consecutive even numbers that are powers of 2 are 2 and 4, the triple must be in the form $2^k - 1$, 2^k and $2^k + 1$. As one of three consecutive integers is always divisible by 3, one of the numbers $2^k - 1$ and $2^k + 1$ must be a power of 3. Consider these two cases separately.

1) Say $3^n = 2^k - 1$; consider the sides of the equation modulo 8: $2^k - 1 \equiv 7$, but $3^{2x} \equiv 1$ and $3^{2x+1} \equiv 3$, a contradiction.

2) Say $3^n = 2^k + 1$; consider the sides of the equation modulo 4: $2^k + 1 \equiv 1$ and $3^{2x} \equiv 1$, $3^{2x+1} \equiv 3$ implying that n is even. Now consider the sides of the equation modulo 7: since $2^{3y} + 1 \equiv 2$, $2^{3y+1} + 1 \equiv 3$, $2^{3y+2} + 1 \equiv 5$ and $3^{6z} \equiv 1$, $3^{6z+2} \equiv 2$, $3^{6z+4} \equiv 4$ at the same time, we must have $2^k + 1 = 3^n \equiv 2 \pmod 7$.

Consequently the first number of the triple $2^k - 1$ is divisible by 7, therefore it is a power of 7: $2^k - 1 = 7^m$. Consider the sides of the equation modulo 16: $2^k - 1 \equiv 15$, but $7^{2w} \equiv 1$, $7^{2w+1} \equiv 7$ — a contradiction.
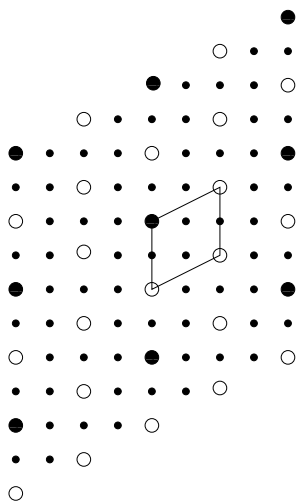

Figure 4

S4. *Answer:* $n = 4$.

Solution 1. It is evident that one cannot find suitable colouring in the case $n = 2$ (one can only colour the points on the straight line determined by the two initial points) and that suitable colouring exists in the case $n = 4$ (e.g. colour points $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$).

It remains to prove that no such colouring exists in the case $n = 3$. If one chooses three points collinear, then no points outside the common straight line can be coloured white. Now we prove that if one chooses three points at three vertices of a parallelogram (such parallelogram always exists if the points are not chosen collinear), one cannot colour the fourth vertex (having evidently integral coordinates) of this parallelogram. For this it suffices to show such subset of all points of integral coordinates that is symmetric in respect of any of its point and contains the three vertices of the parallelogram, but does not include the fourth vertex. We obtain such set of points in the following way:

(1) Consider a lattice on the plane, the axes of the lattice being determined by the edges of the parallelogram, one lattice-point in the opposite vertex of the un-coloured vertex and the distance between lattice points being equal to twice the respective side-length of the parallelogram.

(2) Choose the lattice-points and the midpoints of segments that connect the neighbouring lattice points and are parallel to the axes of the lattice (these points are shown white in figure 4).

Remark. The construction described above can be presented more conveniently if one performs such affine transduction that takes the three vertices of the parallelogram to points $(0,0)$, $(0,1)$ and $(1,0)$ — then the points to be chosen into the subset are exactly the points at least one of the coordinates of which is an even number.

Solution 2. Similarly to the previous solution we notice that it suffices to find such a set of points of integral coordinates on the plane that:

(i) is symmetrical in respect of every its point (i.e. any point that is symmetrical to any point in the set in respect to any other point in the set also belongs to the set);

(ii) contains all three initial white points;

(iii) does not contain all points of integral coordinates on the plane.

In order to construct such a set we consider the parities of the three initial points and see that one can (if necessary, by shifting the coordinate system by one unit) always obtain the situation where all of these three points have at least one of the coordinates an even number. Now the set in question can be expressed by all of these points of integral coordinates on the plane that have at least one of the coordinates an even number (if two points of integral coordinates are symmetrical in respect to some third point of integral coordinates, then the respective coordinates of the two points are of the same parity; the conditions (ii) and (iii) are evidently satisfied).

S5. *Answer:* b) $2^{(K-1)(P-1)}$.

Solution 1. a) Consider the situation described in the problem as a table consisting of K rows and P columns, corresponding respectively to the newspapers and the politicians, and colour the square in the ith row and the jth column black in the case the ith newspaper wrote about the jth politician. Let a_i and b_j be respectively the number of coloured squares in the ith row and in the jth column. For $a_1 + \ldots + a_K = b_1 + \ldots + b_P$ (either sum represents the total number of coloured squares) and according to the conditions of the problem all the addends are odd, the counts of the addends K and P must be of the same parity.

b) The number of the squares not being in the last row and in the last column is $(K-1)(P-1)$, and as we can either colour or leave uncoloured any of these squares, the total number of possibilities to colour these squares is $2^{(K-1)(P-1)}$. Show that any such colouring can be completed exactly uniquely to the colouring of the whole table satisfying the conditions of the problem (i.e. the number of coloured squares in every row and in every column is odd). Indeed: the square in the last column must be coloured exactly in these rows (from the 1st until the $(K-1)$th) that have an even number of squares coloured. After that the square in the last row must be coloured exactly in these columns (from the 1st until the Pth) that have an even number of squares coloured. It remains to check that the last row has an odd number of squares coloured as well — this is proved by the

7

8

fact that the numbers of rows and columns are by a) odd, all the rows contain altogether as many coloured squares as do all the columns, and all the rows except the last and all the columns contain an odd number of coloured squares.

Solution 2. a) Consider the situation described in the problem as a graph of K yellow and P brown vertices, where one has edges only between vertices of different colours; an edge between the ith yellow and the jth brown vertex shows that the ith newspaper wrote about the jth politician. According to the conditions of the problem an odd number of edges emerges from every vertex of the graph, the sum of all these numbers equals twice the number of edges (as every edge counts at its both vertex) and is thus an even number. Hence the number of vertices $K + P$ is an even number, implying that K and P are of the same parity.

b) Let the yellow vertices be $A_1, \ldots, A_K$ and the brown vertices $B_1, \ldots, B_P$. One has $2^{(K-1)(P-1)}$ possibilities for joining the vertices $A_1, \ldots, A_{K-1}$ and $B_1, \ldots, B_{P-1}$ (one may or may not have an edge between the vertices in any of the pairs (A_i, B_j)). We shall prove that every such graph can be completed to a graph satisfying the conditions of the problem (i.e. an odd number of edges emerging from every vertex) by adding edges that emerge from A_K and B_K in an exactly unique way. Indeed, one must draw edges from the vertex A_K exactly into the vertices B_j ($j = 1, \ldots, P-1$) that had an even number of edges emerging. After that one must draw edges from the vertex B_P exactly into the vertices A_i ($i = 1, \ldots, K$) that had an even number of edges emerging. It remains to check that the vertex B_P now also has an odd number of edges — this is proved by the fact that according to a) the total number of vertices is even, all the rest of the vertices has an odd number of edges emerging and the total sum of edges emerging from any vertex is twice the number of edges, thus an even number.

Spring Open Contest: February 2003

Juniors (up to 10th grade)

1. A four-digit number a not containing digit 9 is a square of an integer. If we increase every digit of a by 1, we obtain a square of an integer again. Find all 4-digit numbers with such property.

2. The shape of a dog kennel from above is an equilateral triangle with side length 1 m and its corners in points A, B and C, as shown in the picture. The chain of the dog is of length 6 m and its end is fixed to the corner in point A. The dog himself is in point K in a way that the chain is tight and points K, A and B are on the same straight line. The dog starts to move clockwise around the kennel, holding the chain tight all the time. How long is the walk of the dog until the moment when the chain is tied round the kennel at full?

3. A car, a motorcycle, a moped and a bicycle were driving at unvaried speeds in a straight road. At 12:00 the car passed the moped driving in the same direction and met the bicycle and the motorcycle driving in the opposite direction respectively at 14:00 and 16:00. The motorcycle met the moped at 17:00 and passed the bicycle at 18:00. At what time did the moped meet the bicycle?

4. Consider the points A_1 and A_2 on the side AB of the square $ABCD$ taken in such a way that $|AB| = 3\,|AA_1|$ and $|AB| = 4\,|A_2B|$, similarly consider points B_1 and B_2, C_1 and C_2, D_1 and D_2 respectively on the sides BC, CD and DA. The intersection point of straight lines D_2A_1 and A_2B_1 is E, the intersection point of straight lines A_2B_1 and B_2C_1 is F, the intersection point of straight lines B_2C_1 and C_2D_1 is G and the intersection point of straight lines C_2D_1 and D_2A_1 is H. Find the area of the square $EFGH$, knowing that the area of $ABCD$ is 1.

5. Is it possible to write one of the letters A, B, C or D in every square of an infinite checked paper in such a way that every 2×2 square contains all four letters?

Seniors (11th and 12th grade)

1. Find the values of sharp angles α and β that satisfy

$$\begin{cases} \cos^2\alpha + \cos^2\beta = \dfrac{3}{2} \\ \sin\alpha \cdot \sin\beta = \dfrac{1}{4} \end{cases}.$$

2. Juku has 2003 wooden sticks (nothing is known about their lengths). Juku constructs a rectangle of all these sticks, breaking some sticks into two parts. Find the least number of breakings that would be sufficient for any lengths of the sticks.

3. The sequence $\{F_n\}$ is defined as follows:

$$F_1 = 1, \quad F_2 = a, \quad F_n = F_{n-1} + F_{n-2}, \text{ if } n \geqslant 3.$$

a) Do there exist integers a and $N > 1$ that no member of the sequence $\{F_n\}$ would be divisible by N?

b) Do there exist integers a and $N > 1$ that some two consecutive members of the sequence $\{F_n\}$ would be divisible by N?

4. Consider the points D, E and F on the respective sides BC, CA and AB of the triangle ABC in a way that the segments AD, BE and CF have a common point P. Let $\dfrac{|AP|}{|PD|} = x$, $\dfrac{|BP|}{|PE|} = y$ and $\dfrac{|CP|}{|PF|} = z$. Prove that $xyz - (x + y + z) = 2$.

5. Consider $n \times n$ squares painted black and white as a chessboard in a computer monitor. One can mark any rectangle consisting of whole squares by mouse-dragging and invert the colours of the squares in a marked rectangle by mouse-clicking. Find the smallest number of mouse-clicks by what one can convert all the squares monochrome.
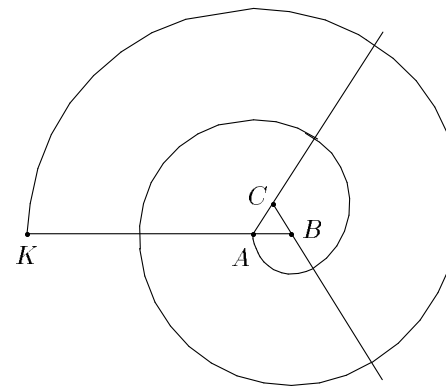
Solutions of Spring Open Contest

J1. *Answer:* 2025 is the only such number.

Solution. Let $a = x^2$ and $a + 1111 = y^2$. Then

$$1111 = 101 \cdot 11 = y^2 - x^2 = (y + x)(y - x)$$

that implies $y + x = 101$, $y - x = 11$ or $y + x = 1111$, $y - x = 1$. The first case leads to $x = 45$, $y = 56$ and $x^2 = 2025$. The second case leads to $x = 555$ and $y = 556$, but the squares of these numbers are not 4-digit numbers.

J2. *Answer:* 14π m.

Solution: The dog's path consists of six $120°$ arcs of a circle, the radii of which are 6, 5, 4, 3, 2 ja 1 metres (see figure 5). Hence the total length of the dog's path is

$$\frac{1}{3} \cdot 2\pi \cdot (6 + 5 + 4 + 3 + 2 + 1) = \frac{2\pi}{3} \cdot 21 = 14\pi \text{ (m)}.$$

Figure 5

J3. *Answer:* At 15.20.

Solution 1. Construct a diagram with the graphs of movement of the car, the motorcycle, the moped and the bicycle, and have one axis for time in hours and the other for distance in kilometers. Let A, B, C, D, E and F be points in this diagram that respectively correspond to the meeting of the car and the moped, the car and the bicycle, the car and the motorcycle, the motorcycle and the moped, the motorcycle and the bicycle, and the moped and the bicycle (see figure 6). In this case B is the midpoint of segment AC and D is the midpoint of segment CE. Hence F is the intersection point of the medians AD and EB in triangle ACE, implying that $|AF| = \dfrac{2}{3}|AD|$ and the difference in "time coordinates" between points A and F is $\dfrac{2}{3} \cdot (17 - 12) = \dfrac{10}{3}$ hours. Therefore the moped met the bicycle $\dfrac{10}{3}$ hours, or 3 hours and 20 minutes later than it met the car, i.e. at 15.20.
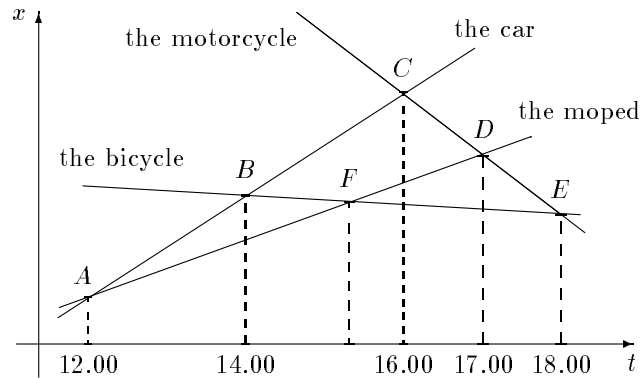
Figure 6

Solution 2. Say "zero time" and "zero point" the time and place of the meeting of the car and the moped, and let the positive direction of the axis be the direction where the car and the moped move. Let the speeds of the car, the motorcycle, the moped and the bicycle be respectively a, b, c and d, then at time t their respective locations are at, $4a - b(t-4)$ (since the car and the motorcycle met at time $t=4$), ct and $2a - d(t-2)$ (since the car and the bicycle met at time $t=2$). As the motorcycle and the moped met at time $t=5$, we obtain an equation

$$4a - b = 5c, \tag{1}$$

and from the meeting of the motorcycle and the bicycle at time $t=6$ we get

$$4a - 2b = 2a - 4d. \tag{2}$$

What should be computed is the meeting time of the moped and the bicycle, i.e. the time x that satisfies $cx = 2a - d(x-2)$, or $x = 2\dfrac{a+d}{c+d}$. Having expressed b from the equation (1) and substituted it into the equation (2), we obtain $4a - 2(4a - 5c) = 2a - 4d$, or $6a = 10c + 4d$. Adding to both sides $6d$ gives $6(a+d) = 10(c+d)$, implying $x = 2\dfrac{a+d}{c+d} = 2 \cdot \dfrac{10}{6} = \dfrac{10}{3}$. Therefore the moped met the bicycle $\dfrac{10}{3}$ hours or 3 hours 20 minutes later than it met the car, i.e. at 15.20.
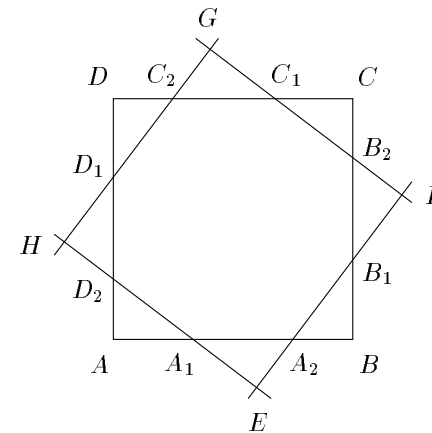

Figure 7

J4. Right triangles AA_1D_2 and EA_1A_2 are similar, since $\angle AA_1D_2 = \angle EA_1A_2$ (see figure 7). We shall show that $|D_2A_1| = |A_1A_2|$, i.e. these triangles are congruent. Indeed,

$$|D_2A_1| = \sqrt{|D_2A|^2 + |AA_1|^2} = \sqrt{\frac{1}{16}|AB|^2 + \frac{1}{9}|AB|^2} = \frac{5}{12}|AB|$$

and

$$|A_1A_2| = |AB| - |AA_1| - |A_2B| = \left(1 - \frac{1}{3} - \frac{1}{4}\right)|AB| = \frac{5}{12}|AB|.$$

Therefore $|D_2A_1| = |A_1A_2|$, $|AA_1| = |A_1E|$, $|A_2B| = |D_2A| = |EA_2| = |HD_2|$ and

$$|HE| = |HD_2| + |D_2A_1| + |A_1E| = |A_2B| + |A_1A_2| + |AA_1| = |AB|.$$

Consequently the squares $ABCD$ and $EFGH$ are of the same side length, i.e. the area of the square $EFGH$ is 1 as well.

J5. *Answer:* no.

Solution. Assume the required configuration of the letters exists. Then no row can contain two similar letters consecutively, therefore one must have three different letters consecutively — without loss of generality assume they be A, B and C. The next row must contain aligned to them the letters C, D and A, the row after the next again has A, B and C, etc. We see that any of the three columns contains only two letters, a contradiction with the assumption made above.

S1. *Answer:* $\alpha = \beta = 30°$.

13

14

Substitutions $\cos^2 \alpha = 1 - \sin^2 \alpha$ and $\cos^2 \beta = 1 - \sin^2 \beta$ give a system of equation similar to the given system

$$\begin{cases} \sin^2 \alpha + \sin^2 \beta = \dfrac{1}{2} \\ \sin \alpha \cdot \sin \beta = \dfrac{1}{4} \end{cases}.$$

By letting $x = \sin \alpha$ and $y = \sin \beta$, one has a system of equations in x and y

$$\begin{cases} x^2 + y^2 = \dfrac{1}{2} \\ x \cdot y = \dfrac{1}{4} \end{cases}.$$

Having added two times the second equation to the first, we obtain $(x+y)^2 = \dfrac{1}{2} + 2 \cdot \dfrac{1}{4} = 1$. Since α and β are acute angles, x and y are positive numbers, therefore $x + y = 1$, or $y = 1 - x$. Substituting it into the second equation in the system, we obtain $x \cdot (1 - x) = \dfrac{1}{4}$ or

$$0 = x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2,$$

from which $x = \dfrac{1}{2}$ and $\alpha = 30°$. The equation $x + y = 1$ now gives $y = \dfrac{1}{2}$ and $\beta = 30°$.

S2. *Answer:* 2.

Solution. Juku may cut one stick into two halves; consider these halves one pair of sides of a rectangle and divide the rest of the sticks into two sets A and B. If the total lengths of the sticks in sets A and B are equal, no more sticks need to be broken — but if, for instance, the total length of the sticks in A is bigger, we put one by one sticks from A to B until the inequality of total lengths reverses. Now it suffices to break the last stick that was put from A to B (in the case the total lengths equalized no breaking is necessary).

One does not find sufficient to break only 1 stick e.g. in the case where the lengths are $1, 2, \ldots, 2^{2002}$. If it would be enough to break only one stick in order to construct a rectangle, one had its two sides of integer and two sides of fractional lengths (if all four sides were of integral length, their sum would be an even number, but $1 + 2 + \ldots + 2^{2002}$ is odd). Hence one pair of opposite sides should be constructed only of initial sticks, but it is not possible, since one can find a stick longer than the total sum of the others in every subset of the initial sticks.

S3. *Answer:* a) yes; b) no.

a) Having $a = 3$, the residues of F_n modulo $N = 5$ are the following

$$1, 3, 4, 2, 1, 3, \ldots .$$

Since every term in the sequence is determined by two previous terms, the residues modulo 5 are also determined by two previous residues, hence the quadruple of residues $1, 3, 4, 2$ will repeat infinitely and consequently no member of the sequence is divisible by 5.

b) Having written $F_n = F_{n-1} + F_{n-2}$ in the form $F_{n-2} = F_n - F_{n-1}$, we see that every term of this sequence equals the difference of two next consecutive terms. Therefore if some two consecutive terms of the sequence were divisible by $N > 1$, all the terms preceeding them would also be divisible by N, including the first term $F_1 = 1$ — a contradiction. Consequently for any choice of a two consecutive terms of the sequence are pairwise primes.

S4. Let the base points of heights drawn to the segment AB in triangles ABC and ABP be respectively K and L (one or both of these points may lie on the extension of side AB) and let Q be the intersection point of the segment CK with straight line s, passing through point P and being parallel to side AB (see figure 8). Let S_{XYZ} be the area of triangle XYZ, then

$$\frac{|PF|}{|CF|} = \frac{|QK|}{|CK|} = \frac{|PL|}{|CK|} = \frac{S_{ABP}}{S_{ABC}}$$

and similarly $\dfrac{|PD|}{|AD|} = \dfrac{S_{BCP}}{S_{ABC}}$ and $\dfrac{|PE|}{|BE|} = \dfrac{S_{ACP}}{S_{ABC}}$. Since

$$\frac{|PD|}{|AD|} = \frac{1}{1 + \dfrac{|AP|}{|PD|}} = \frac{1}{1 + x}$$

and similarly $\dfrac{|PE|}{|BE|} = \dfrac{1}{1+y}$ and $\dfrac{|PF|}{|CF|} = \dfrac{1}{1+z}$, we have

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = \frac{S_{BCP} + S_{ACP} + S_{ABP}}{S_{ABC}} = 1,$$

or $(1+y)(1+z) + (1+x)(1+z) + (1+x)(1+y) = (1+x)(1+y)(1+z)$, giving after multiplication and collecting similar terms the required equation $xyz - (x+y+z) = 2$.
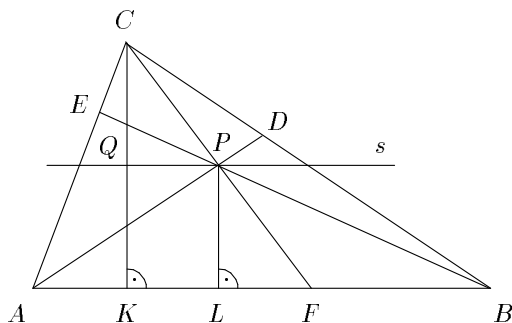
Figure 8

S5. *Answer:* n, if n is even; $n - 1$, if n is odd.

Solution. Every mouse-click can convert monochrome at most 4 pairs of squares of different colours on the outer edge of the rectangle. As one has $4(n-1)$ of such pairs initially, one requires at least $n-1$ mouse-clicks. For even n one needs at least n mouse-clicks, since the corner squares are not of same colour and by marking any rectangle containing a corner square one converts monochrome only 2 pairs of squares of different colours on the outer edge of the rectangle.

These numbers prove to be sufficient. We may choose subsequently the 2nd, 4th, 6th, ... row and then the 2nd, 4th, 6th, ... column — for odd n we have altogether $2 \cdot \dfrac{n-1}{2} = n - 1$ mouse-clicks and for even n we have altogether $2 \cdot \dfrac{n}{2} = n$ mouse-clicks.

Final Round of National Olympiad: March 2003

9th grade

1. Let A_1, A_2, ..., A_m and B_2, B_3, ..., B_n be the points on a circle such that $A_1 A_2 \ldots A_n$ is a regular m-gon and $A_1 B_2 \ldots B_n$ is a regular n-gon whereby $n > m$ and the point B_2 lies between A_1 and A_2. Find $\angle B_2 A_1 A_2$.

2. Find all positive integers n such that

$$ n + \left[\frac{n}{6}\right] \neq \left[\frac{n}{2}\right] + \left[\frac{2n}{3}\right]. $$

   Here $[x]$ denotes the largest integer not greater than x.

3. In the rectangle $ABCD$ with $|AB| < 2|AD|$, let E be the midpoint of AB and F a point on the chord CE such that $\angle CFD = 90°$. Prove that FAD is an isosceles triangle.

4. Ella the Witch was mixing a magic elixir which consisted of three components: 140 ml of reindeer moss tea, 160 ml of fly agaric extract, and 50 ml of moonshine. She took an empty 350 ml bottle, poured 140 ml of reindeer moss tea into it and started adding fly agaric extract when she was disturbed by its black cat Mefisto. So she mistakenly poured too much fly agaric extract into the bottle and noticed her fault only later when the bottle filled before all 50 ml of moonshine was added. Ella made quick calculations, carefully shaked up the contents of the bottle, poured out some part of liquid and added some amount of mixture of reindeer moss tea and fly agaric extract taken in a certain proportion until the bottle was full again and the elixir had exactly the right compositsion. Which was the proportion of reindeer moss tea and fly agaric extract in the mixture that Ella added into the bottle?

5. Is it possible to cover an $n \times n$ chessboard which has its center square cut out with tiles shown in the picture (each tile covers exactly 4 squares; tiles can be rotated and turned around) if a) $n = 5$; b) $n = 2003$?
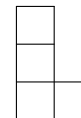
10th grade

1. The picture shows 10 equal regular pentagons where each two neighbouring pentagons have a common side. The smaller circle is tangent to one side of each pentagon and the larger circle passes through the opposite vertices of these sides. Find the area of the larger circle if the area of the smaller circle is 1.
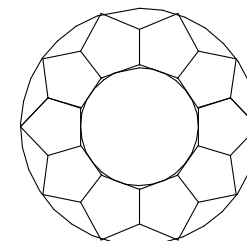
2. Find all possible integer values of $\dfrac{m^2 + n^2}{mn}$ where m and n are integers.

3. In the acute-angled triangle ABC all angles are greater than $45°$. Let AM and BN be the heights of this triangle and let X and Y be the points on MA and NB, respecively, such that $|MX| = |MB|$ and $|NY| = |NA|$. Prove that MN and XY are parallel.

4. Let a, b, and c be positive real numbers not greater than 2. Prove the inequality

$$ \frac{abc}{a + b + c} \leqslant \frac{4}{3}. $$

5. The game *Clobber* is played by two on a strip of $2k$ squares. At the beginning there is a piece on each square, the pieces of both players stand alternately. At each move the player shifts one of his pieces to the neighbouring square that holds

a piece of his opponent and removes his opponent's piece from the table. The moves are made in turn, the player whose opponent cannot move anymore is the winner.

Prove that if for some k the player who does not start the game has the winning strategy, then for $k+1$ and $k+2$ the player who makes the first move has the winning strategy.

11th grade

1.

Juhan is touring in Europe. He stands on a highway and watches cars. There are three cars driving along the highway at constant speeds: an Opel and a Trabant in one direction and a Mercedes in the opposite direction. At the moment when the Trabant passes Juhan, the Opel and the Mercedes lie at equal distances from him in opposite directions. At the moment when the Mercedes passes Juhan, the Opel and the Trabant lie at equal distances from him in opposite directions. Prove that at the moment when the Opel passes Juhan, also the Mercedes and the Trabant lie at equal distances from him in opposite directions.

2. Prove that for all positive real numbers a, b, and c

$$\sqrt[3]{abc} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant 2\sqrt{3}\,.$$

When does the equality occur?

3. Let ABC be a triangle and A_1, B_1, C_1 points on BC, CA, AB, respectively, such that the lines AA_1, BB_1, CC_1 meet at a single point. It is known that A, B_1, A_1, B are concyclic and B, C_1, B_1, C are concyclic. Prove that a) C, A_1, C_1, A are concyclic; b) AA_1, BB_1, CC_1 are the heights of ABC.

4. Prove that there exist infinitely many positive integers n such that $\sqrt{n}$ is not an integer and n is divisible by $[\sqrt{n}]$. (Here $[x]$ denotes the largest integer not greater than x.)

5. For which positive integers n is it possible to cover a $(2n+1) \times (2n+1)$ chessboard which has one of its corner squares cut out with tiles shown in the figure (each tile covers exactly 4 squares; tiles can be rotated and turned around)?
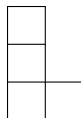
12th grade

1. Jüri and Mari both wish to tile an $n \times n$ chessboard with cards shown in the picture (each card covers exactly one square). Jüri wants that for each two cards that have a common edge, the neighbouring parts are of different color, and Mari wants that the neighbouring parts are always of the same color. How many possibilities does Jüri have to tile the chessboard and how many possibilities does Mari have?

2. Solve the equation $\sqrt{x} = \log_2 x$.

3. Let ABC be a triangle with $\angle C = 90°$ and D a point on the ray CB such that $|AC| \cdot |CD| = |BC|^2$. A parallel line to AB through D intersects the ray CA at E. Find $\angle BEC$.

4. Call a positive integer *lonely* if the sum of reciprocals of its divisors (including 1 and the integer itself) is not equal to the sum of reciprocals of divisors of any other positive integer. Prove that a) all primes are lonely; b) there exist infinitely many non-lonely positive integers.

5. On a lottery ticket a player has to mark 6 numbers from 36. Then 6 numbers from these 36 are drawn randomly and the ticket wins if none of the numbers that came out is marked on the ticket. Prove that a) it is possible to mark the numbers on 9 tickets so that one of these tickets always wins; b) it is not possible to mark the numbers on 8 tickets so that one of tickets always wins.

Solutions of Final Round

9-1. *Answer:* $\dfrac{\pi}{m} - \dfrac{\pi}{n}$.

Let O be the center of the circle. Then $\angle B_2 A_1 A_2 = \angle B_2 O A_2 / 2 = (\angle A_1 O A_2 - \angle A_1 O B_2)/2 = \pi/m - \pi/n$.

9-2. *Answer:* $n = 6k + 1$ where $k = 0, 1, \ldots$

If the equality holds (does not hold) for some n, then it holds (does not hold) for $n + 6$. Indeed, $(n + 6) + \left[\dfrac{n+6}{6}\right] = n + \left[\dfrac{n}{6}\right] + 7$ and $\left[\dfrac{n+6}{2}\right] + \left[\dfrac{2(n+6)}{3}\right] = \left[\dfrac{n}{2}\right] + \left[\dfrac{2n}{3}\right] + 7$. Therefore it suffices to check the values $n = 0, 1, 2, 3, 4, 5$. Doing so, we find that $n = 1$ is the only case when the equality is violated.

9-3. Since E is the midpoint of AB, the right-angled triangles EAD and EBC are congruent. Also, since $\angle DAE = \angle DFE = 90°$, the quadrilateral $AEFD$ is cyclic. Therefore $\angle AFD = \angle AED = \angle BEC$. On the other hand, $\angle ADF = 180° - \angle AEF = \angle BEC$. Hence $\angle AFD = \angle ADF$.

Solution 2. Let K be the intersection of CE and DA. Then EA is the midline of KCD, therefore $AK = AD$. Since $\angle KFD = 90°$, the points K, F, and D lie on the circle centered at A. This means $AF = AD$.

9-4. *Answer:* $2 : 3$.

When the bottle got filled first time, the reindeer moss tea made up 140 ml of its 350 ml volume. Therefore the mixture that was poured out consisted of $2/5$ of reindeer moss tea and $3/5$ of other substances. Since the amount of liquid that was poured out of the bottle and the amount that was added afterwards are equal and the bottle finally contained 140 ml of reindeer moss tea again, also the mixture that was added was made up of exactly $2/5$ of reindeer moss tea. So the amount of moonshine in that mixture was $3/5$.

9-5. *Answer:* a) yes; b) yes.

a) See figure 9.

b) Figure 10 shows how to tile the border of width 2 around the $(4k-1) \times (4k-1)$ square (we use the fact that 2×4 rectangle can be covered with two tiles). Starting from the tiling for $n = 3$ (its existence can be seen from the figure 9) and applying this construction 500 times, we get the tiling for $n = 2003$.
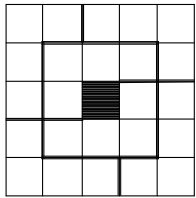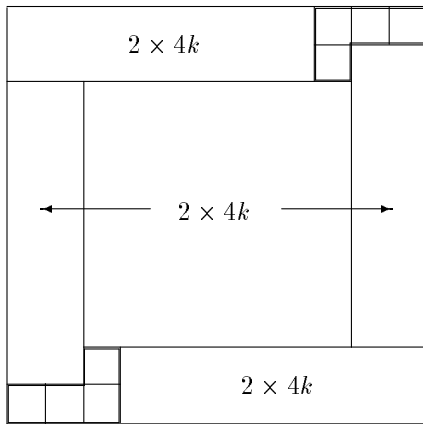

Figure 9   Figure 10

10-1. *Answer:* 4.

Let O be the common center of the circles and $ABCDE$ one of the pentagons whereby the inner circle touches the side AB and the outer circle passes through D. Let Q be the center of $ABCDE$. Then $\angle AOB = 360°/10 = 36°$ and $\angle ADB = \angle AQB/2 = 360°/10 = 36°$. So the isosceles triangles AOB and ADB are congruent and the radius of the inner circle is exactly half of the radius of the outer circle.

10-2. *Answer:* 2 and -2.

Using the fact that mn divides $m^2 + n^2$, we conclude that m divides n^2 and n divides m^2. Therefore m and n have the same prime divisors. Now let $m = p^a m'$ and $n = p^b n'$ where the prime p is not a divisor of m' and n'. If for example $a > b$, then mn and m^2 are both divisible by p^{a+b} but n^2 is not (it is only divisible by p^{2b} where $2b < a + b$). That is, $(m^2 + n^2)/(mn)$ is not an integer, a contradiction. So it must be $a = b$ and consequently $m = \pm n$. Then $$\frac{m^2 + n^2}{mn} = \frac{2n^2}{\pm n^2} = \pm 2.$$

10-3. The quadilateral $AXYB$ is cyclic since $\angle AXB = 180° - \angle MXB = 135°$ and analogously $\angle AYB = 135°$. Also, the quadrilateral $ANMB$ is cyclic. Therefore $\angle MXY = 180° - \angle AXY = \angle ABN = \angle AMN$ which means that MN and XY are parallel.

10-4. First, from the inequality between arithmetic mean and geometric mean we get $a + b + c \geqslant 3\sqrt[3]{abc}$. Second, since $a \leqslant 2$, $b \leqslant 2$, and $c \leqslant 2$, we have $abc \leqslant 8$. Now $$\frac{abc}{a + b + c} \leqslant \frac{abc}{3\sqrt[3]{abc}} = \frac{\sqrt[3]{(abc)^2}}{3} \leqslant \frac{\sqrt[3]{64}}{3} = \frac{4}{3}.$$

10-5. If the length of the strip is $2(k + 1)$, then at his first move, the first player beats his opponent's piece that stands at the end of the strip. This divides the pieces into two sections: one of length $2k$ and the other of length 1. Since the second section cannot change, the situation is equivalent to playing the game on a strip of length $2k$ where the second player makes the first move. So the first player can follow the winning strategy.

If the length of the strip is $2(k+2)$, then at his first move, the first player beats his opponent's piece that stands on the third square from the end with his piece that stands on the fourth square from the end. After that the board again contains two sections: one of length $2k$ and the other of length 3. If the second player makes a move in the first section, the first player responds according to the winning strategy. If the second player makes a move in the second section, the first player also makes his move in the second section which thereafter has only one piece left.

11-1. Let t_1 and t_2 be the time instants when the Trabant and the Mercedes passed Juhan, respectively. The Mercedes was at t_1 some distance d away from Juhan

and by t_2 it reached him. The Opel also was at t_1 the distance d away from Juhan (on the other side) and at t_2 its distance to Juhan was the same as the distance the Trabant covered between t_1 and t_2. This means that the velocity of the Mercedes equals the sum of velocities of the Opel and the Trabant. If t_3 is the time instant when the Opel reaches Juhan, having covered the distance d since t_1, the Mercedes has covered the distance d, passing Juhan at t_2 in the opposite direction, plus the distance that the Trabant has covered between t_1 and t_3.

11-2. *Answer:* equality occurs when $a = b = c = \sqrt{3}$.

Using twice the inequality between arithmetic and geometric means, we get

$$\sqrt[3]{abc} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant \sqrt[3]{abc} + 3\sqrt[3]{\frac{1}{abc}} \geqslant 2\sqrt{\sqrt[3]{abc} \cdot 3\sqrt[3]{\frac{1}{abc}}} =$$

$$= 2\sqrt{3} \; .$$

The equality holds if and only if $a = b = c$ and $\sqrt[3]{abc} = 3\sqrt[3]{\frac{1}{abc}}$. These conditions

give $\sqrt[3]{a^3} = 3\sqrt[3]{\frac{1}{a^3}}$ or $a = \frac{3}{a}$. Hence $a = b = c = \sqrt{3}$.

11-3. a) Let AA_1, BB_1, and CC_1 intersect at P. From the property of intersecting chords we have $|PA| \cdot |PA_1| = |PB| \cdot |PB_1|$ and also $|PC| \cdot |PC_1| = |PB| \cdot |PB_1|$. So $|PA| \cdot |PA_1| = |PC| \cdot |PC_1|$ from which it follows that the points C, A_1, C_1, and A are concyclic.

b) We know that AB_1A_1B, BC_1B_1C, and CA_1C_1A are cyclic quadrilaterals. Therefore $\angle AB_1B = \angle AC_1C$ because $\angle CB_1B$ and $\angle CC_1B$ are equal. Now $\angle AA_1B = \angle AB_1B = \angle AC_1C = \angle AA_1C$. Hence $\angle AA_1B = 90°$. Analogously $\angle BB_1C = 90°$ and $\angle CC_1A = 90°$.

11-4. The numbers $m^2 + m$ and $m^2 + 2m$ all have the desired property where m is an arbitrary positive integer. Indeed, the numbers $\sqrt{m^2 + m}$ and $\sqrt{m^2 + 2m}$ are not integers, since $m^2 < m^2 + m < m^2 + 2m < (m+1)^2$. In addition, $\left[\sqrt{m^2 + m}\right] = \left[\sqrt{m^2 + 2m}\right] = m$ which is a divisor of both $m^2 + m$ and $m^2 + 2m$.

11-5. *Answer:* all even numbers.

Figure 11 gives the tiling for $n = 2$, figure 12 shows how to extend the tiling for $n = 2m$ to the case $n = 2(m + 1)$ (we use the fact that 2×4 rectangle can be covered with two tiles).

Let now $n = 2m + 1$. Let us color the rows alternatingly black and white. Since there are an even number of columns of full length $4m + 3$ and one column of length $4m + 2$ which contains $2m + 1$ black squares and $2m + 1$ white squares, the board altogether has an odd number of black squares and an odd number of white squares. Each tile, not depending of its position, always covers 3 squares of one color and 1 square of other color, hence the total number of tiles must be odd. On the other hand, the board has $(4m + 3)^2 - 1 = 16m^2 + 24m + 8$ squares. This number is divisible by 8, so the number of tiles must be even, a contradiction.
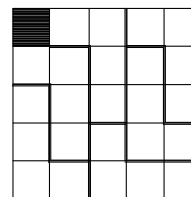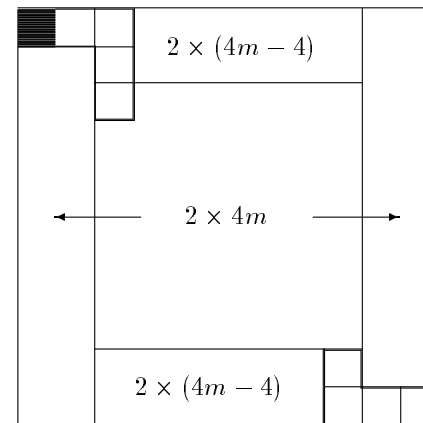

Figure 11   Figure 12

12-1. *Answer:* both Jüri and Mari have 4^n possibilities.

Let us place n cards on the n squares of main diagonal. This can be done in 4^n ways, since each card can be in one of 4 positions. After that, the position of other cards is uniquely determined for both Jüri and Mari: if we know the colors of two neighbouring edges of a square, the card can be placed on that square only in one way. So we can fill the whole board diagonal by diagonal.

12-2. *Answer:* $x = 4$ and $x = 16$.

First note that 4 and 16 satisfy the equation. To show that there are no more solutions, consider the function $f(x) = \sqrt{x} - \log_2 x$. Finding the derivative $f'(x) = \dfrac{\sqrt{x}\ln 2 - 2}{2x\ln 2}$, we see that $f(x)$ has only one extremum point $x = \left(\dfrac{2}{\ln 2}\right)^2$. Because $f(x)$ is continuously differentiable in its domain $(0, \infty)$, it must have a local extremum between each two zero points. Therefore $f(x)$ cannot have more than 2 zero points.

12-3. *Answer:* $45°$.

Triangles EDC and ABC are similar with the similarity ratio $k = |DC|/|BC|$. From the given facts we get $|DC|/|BC| = |BC|/|AC|$.

Now $|EC| = |AC| \cdot k = |AC| \cdot |BC|/|AC| = |BC|$. Triangle BCE is right-angled and isosceles, so $\angle CEB = 45°$.

12-4. a) For a prime p, the sum of reciprocals of its divisors is $(p+1)/p$. If there exists another positive integer $a \neq p$ whose sum of reciprocals of divisors is $(p+1)/p$, then p must be one of a's divisors. Then a is divisible at least by 1, p, and a, and the sum of their reciprocals is greater than $1 + 1/p$.

b) Suppose that there exist an integer a with divisors $d_1, \ldots, d_k$ and a second integer b with divisors $e_1, \ldots, e_l$ such that $1/d_1 + \ldots + 1/d_k = 1/e_1 + \ldots + 1/e_l$. Let p be a prime that is not a divisor of neither a nor b. Then ap has divisors $d_1, \ldots, d_k, d_1 p, \ldots, d_k p$, and bp has divisors $e_1, \ldots, e_l, e_1 p, \ldots, e_l p$. Computing their reciprocal sum, we get for ap and bp, respectively:

$$\frac{1}{d_1} + \ldots + \frac{1}{d_k} + \frac{1}{d_1 p} + \ldots + \frac{1}{d_k p} = \left(1 + \frac{1}{p}\right)\left(\frac{1}{d_1} + \ldots + \frac{1}{d_k}\right)$$

$$\frac{1}{e_1} + \ldots + \frac{1}{e_l} + \frac{1}{e_1 p} + \ldots + \frac{1}{e_l p} = \left(1 + \frac{1}{p}\right)\left(\frac{1}{e_1} + \ldots + \frac{1}{e_l}\right)$$

The results are equal. So if there exist non-lonely integers a and b, then all integers ap and bp where p is a prime not dividing a and b are also non-lonely. Now we can take $a = 6$ and $b = 28$ (in general, any two perfect numbers), then $1 + 1/2 + 1/3 + 1/6 = 2$ and $1 + 1/2 + 1/4 + 1/7 + 1/14 + 1/28 = 2$.

12-5. a) Mark the numbers on 9 tickets as follows:

$(1, 2, 3, 4, 5, 6)$	$(10, 11, 12, 13, 14, 15)$	$(19, 20, 21, 22, 23, 24)$
$(1, 2, 3, 7, 8, 9)$	$(10, 11, 12, 16, 17, 18)$	$(25, 26, 27, 28, 29, 30)$
$(4, 5, 6, 7, 8, 9)$	$(13, 14, 15, 16, 17, 18)$	$(31, 32, 33, 34, 35, 36)$

If none of the tickets in the first column wins, two of the six drawn numbers must belong to $\{1, 2, \ldots, 9\}$. If none of the tickets in the second column wins, two of the six drawn numbers must belong to $\{10, 11, \ldots, 18\}$. If none of the tickets in the last column wins, three of the six drawn numbers must belong to $\{19, \ldots, 36\}$. Since only 6 numbers are drawn, all these conditions cannot be satisfied at the same time.

b) If some number is marked on 3 tickets form 8, then the drawn set of numbers can contain this number and one number from each of 5 remaining tickets, so none of 8 tickets wins. Now suppose that each number is marked on at most 2 tickets. All 8 tickets together have 48 numbers marked but since there are only 36 different numbers, at least 12 numbers must occur twice. Without loss of generality, let they be 1, 2, $\ldots$, 12. Let us take two tickets, say A and B, which both have 1 marked. They contain 10 more numbers, so one of the numbers 2, $\ldots$, 12 is not marked on either of them. Let this number be 12. Then 12 is marked on two

of remaining six tickets, say on C and D. Now the drawn set can contain 1, 12 and one number from each of 4 remaining tickets (not A, B, C, D), and again, none of the 8 tickets wins.

IMO Team Selection Test: May 2003

First Day

1. Two treasure-hunters found a treasure containing coins of value $a_1 < a_2 < \ldots < a_{2003}$ (the quantity of coins of each value is unlimited). The first treasure-hunter forms all the possible sets of different coins containing odd number of elements, and takes the most valuable coin of each such set. The second treasure-hunter forms all the possible sets of different coins containing even number of elements, and takes the most valuable coin of each such set. Which one of them is going to have more money and how much more? (*H. Nestra*)

2. Let n be a positive integer. Prove that if the number $\underbrace{99 \ldots 9}_{n}$ is divisible by n, then the number $\underbrace{11 \ldots 1}_{n}$ is also divisible by n. (*H. Nestra*)

3. Let $\mathbb{N}$ be the set of all non-negative integers and for each $n \in \mathbb{N}$ denote $n' = n+1$. The function $A : \mathbb{N}^3 \to \mathbb{N}$ is defined as follows:
   (i) $A(0, m, n) = m'$ for all $m, n \in \mathbb{N}$;
   (ii) $A(k', 0, n) = \begin{cases} n, & \text{if } k = 0, \\ 0, & \text{if } k = 1, \\ 1, & \text{if } k > 1 \end{cases}$ for all $k, n \in \mathbb{N}$;
   (iii) $A(k', m', n) = A(k, A(k', m, n), n)$ for all $k, m, n \in \mathbb{N}$.

   Compute $A(5, 3, 2)$. (*H. Nestra*)

Second Day

4. A deck consists of 2^n cards. The deck is shuffled using the following operation: if the cards are initially in the order

   $$a_1, a_2, a_3, a_4, \ldots, a_{2^n - 1}, a_{2^n},$$

   then after shuffling the order becomes

   $$a_{2^{n-1}+1}, a_1, a_{2^{n-1}+2}, a_2, \ldots, a_{2^n}, a_{2^{n-1}}.$$

Find the smallest number of such operations after which the original order of the cards is restored. (*R. Palm*)

5. Let a, b, c be positive real numbers satisfying the condition $\dfrac{1}{ab} + \dfrac{1}{ac} + \dfrac{1}{bc} = 1$.

Prove the inequality

$$\frac{a}{\sqrt{1+a^2}} + \frac{b}{\sqrt{1+b^2}} + \frac{c}{\sqrt{1+c^2}} \leqslant \frac{3\sqrt{3}}{2} \, .$$

When does the equality hold? (*L. Parts*)

6. Let ABC be an acute-angled triangle, O its circumcenter and H its orthocenter. The orthogonal projection of the vertex A to the line BC lies on the perpendicular bisector of the segment AC. Compute $\dfrac{|CH|}{|BO|}$. (*J. Willemson*)

Solutions of Selection Test

1. *Answer:* The first treasure-hunter gets one cheapest coin more than the second one.

Solution. Take all the odd coin sets of the first treasure-hunter and form an even coin set from each one of them by the following rule. If the odd set contains the cheapest coin then leave it out, otherwise add the cheapest coin to the set. This way we obtain exactly all the sets of the second treasure-hunter plus one empty coin set. It is clear that the described operation does not change the most valuable coins of the sets except for the set having originally only one cheapest coin. Hence, the first treasure-hunter gets one cheapest coin more than the second one.

2. The condition of $n \mid \underbrace{11\ldots1}_{n}$ is equivalent to $9n \mid \underbrace{99\ldots9}_{n}$. We will prove a more general result for any positional number system. Namely, we will show by induction on n that for any positive integers n and b the condition $n \mid b^n - 1$ implies $(b-1)n \mid b^n - 1$. Taking $b = 10$ gives the desired result.

For $n = 1$ the claim is true as $b^n - 1 = (b-1)(b^{n-1} + \ldots + b + 1)$. Now assume that for all numbers less than n and any b the claim holds. Take any positive integer b and assume $n \mid b^n - 1$. Consider two possible cases.

a) If n and $b - 1$ are coprime, the conditions $n \mid b^n - 1$ and $b - 1 \mid b^n - 1$ imply $(b-1)n \mid b^n - 1$.

b) If n and $b - 1$ have a common prime factor p, let $n = mp$; then $b^n - 1 = b^{mp} - 1 = (b^p)^m - 1$. As $m < n$, we can use the induction hypothesis for m: as $(b^p)^m - 1 = b^n - 1$ is divisible by $n = mp$ and hence also by m, we have $(b^p - 1)m \mid (b^p)^m - 1 = b^n - 1$. Since $p \mid b - 1$ we get

$b \equiv 1 \bmod p$ and $1 + b + \ldots + b^{p-1} \equiv \underbrace{1 + 1 + \ldots + 1}_{p} \equiv 0 \bmod p$. Hence, $(b^p - 1)m = (b-1)m \cdot (1 + b + \ldots + b^{p-1})$ is divisible by $(b-1)mp = (b-1)n$ and thus $(b-1)n \mid b^n - 1$ as required.

3. *Answer:* 65536.

Solution. First we show by induction on k that for all integers $k > 1$ and any n the equality $A(k, 1, n) = n$ holds. Indeed, if $k = 2$, we have

$$A(2, 1, n) = A(1', 0', n) = A(1, A(1', 0, n), n) =$$
$$= A(1, 0, n) = A(0', 0, n) = n \, .$$

Assuming now that $A(k, 1, n) = n$ holds for some $k > 1$. Then

$$A(k', 1, n) = A(k', 0', n) = A(k, A(k', 0, n), n) = A(k, 1, n) = n \, ,$$

hence we have the required equality for $k' = k + 1$ and the induction is complete. In a similar way we can use induction on m to prove that for any natural numbers m and n the equalities $A(1, m, n) = m + n$, $A(2, m, n) = mn$ and $A(3, m, n) = n^m$ hold.

Next we use induction on k to show that for any integer $k > 0$ the equality $A(k, 2, 2) = 4$ holds. Indeed: for $k = 1$ we have $A(1, 2, 2) = 2 + 2 = 4$. If $A(k, 2, 2) = 4$ for some $k > 0$ we get $k' > 1$ and

$$A(k', 2, 2) = A(k', 1', 2) = A(k, A(k', 1, 2), 2) = A(k, 2, 2) = 4 \, .$$

We also note that for any natural k

$$A(k', 3, 2) = A(k', 2', 2) = A(k, A(k', 2, 2), 2) = A(k, 4, 2) \, .$$

Finally we compute

$$A(5, 3, 2) = A(4', 3, 2) = A(4, 4, 2) = A(3', 3', 2) = A(3, A(3', 3, 2), 2) =$$
$$= A(3, A(3, 4, 2), 2) = 2^{2^4} = 65536 \, .$$

4. *Answer:* At least $2n$ shufflings are needed.

Solution. Assume the cards are labeled by numbers $1, 2, \ldots, 2^n$. We note that the position of the card number x after the first shuffling is $f(x) = 2x \bmod 2^n + 1$. After kth shuffling its position becomes $f^k(x) = 2^k x \bmod 2^n + 1$. Our task is to find the least number k such that for each x the equality $f^k(x) = x$ holds, or equivalently, $2^k \equiv 1 \bmod 2^n + 1$.

Let $k = 2n$. Since

$$2^{2n} \equiv (2^n + 1)^2 - 2(2^n + 1) + 1 \equiv 1 \bmod 2^n + 1 \, ,$$

then after $2n$th shuffling the order of the cards is restored. Assume that for some $m < 2n$ the order is restored after mth shuffling. In this case we must have $m > n$, since the card number 1 only reaches the position 2^n after nth shuffling. Subtracting now the congruence $2^m \equiv 1 \bmod 2^n + 1$ from the congruence $2^{2n} \equiv 1 \bmod 2^n + 1$, we obtain

$$2^m(2^{2n-m} - 1) \equiv 0 \bmod 2^n + 1 \,.$$

This is the contradiction as 2^m and the modulus $2^n + 1$ are coprime and the value of $2^{2n-m} - 1$ is less than the same modulus because of $m > n$. Hence, $k = 2n$ is the least number of shufflings restoring the original order of cards

5. *Answer:* the equality holds iff $a = b = c = \sqrt{3}$.

   Solution. The given equality is equivalent to $abc = a + b + c$. Since $a, b, c > 0$ we can find such $0 < \alpha, \beta, \gamma < \dfrac{\pi}{2}$ that $a = \tan\alpha$, $b = \tan\beta$, $c = \tan\gamma$. Using the equality $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ repeatedly, we obtain $\tan(\alpha + \beta + \gamma) = 0$ or $\alpha + \beta + \gamma = \pi$. Using now the equality $\dfrac{\tan^2 x}{1 + \tan^2 x} = \sin^2 x$ and taking into account that the angles α, β and γ are acute, we can transform the required inequality to the form

   $$\sin\alpha + \sin\beta + \sin\gamma \leqslant \frac{3\sqrt{3}}{2} \,.$$

   Since $\alpha + \beta + \gamma = \pi$ and the sine function is concave in the segment $[0, \pi]$, we can use Jensen's inequality to obtain

   $$\frac{1}{3}\sin\alpha + \frac{1}{3}\sin\beta + \frac{1}{3}\sin\gamma \leqslant \sin\frac{1}{3}(\alpha + \beta + \gamma) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \,,$$

   which implies the required inequality. We also see that the equality holds iff $\alpha = \beta = \gamma = \dfrac{\pi}{3}$, or $a = b = c = \sqrt{3}$.

6. *Answer:* $\sqrt{2}$.

   Solution. Let A' and B' be the base points of the heights drawn from the vertices A and B, respectively. From the conditions of the problem we conclude that triangle $AA'C$ is an isosceles right triangle with the right angle at the vertex A'. Thus, $\angle A'AC = \angle A'CA = 45°$. Hence the triangle $BB'C$ is also right and isosceles. We have $\angle AOB = 2\angle ACB = 2 \cdot 45° = 90°$ and the triangle AOB is right and isosceles as well. In order to prove that $\dfrac{|CH|}{|BO|} = \sqrt{2}$ it is enough to prove that $|AB| = |CH|$. We will show that the triangles ABB' and HCB' are

congruent. Indeed, $\angle AB'B = 90° = \angle HB'C$, $|BB'| = |B'C|$ and $|AB'| = |B'H|$ (since $AB'H$ is also a right isosceles triangle). This completes the proof.