The Caves of Kabul

Osama is hiding in one of the caves numbered 1 to n. He is very dogmatic. He will move to the next cave in either direction each night. Of course, if he is in a cave at either end, he will have no choice in direction.

George is trying to catch Osama by searching one cave each day, and will find him if he searches the right cave. George is very stupid. He will announce in advance the number of days he will be searching, as well as which cave he will search on each day. You are an advisor to George. Can you help him catch Osama?

Let us consider some simple examples. There is only $n = 1$ cave where Osama the First is hiding. George can simply announce that he will search for one day and he will search Cave #1 on that day. Inspite of his own stupidity, George will catch Osama the First without any help.

Osama the Second is hiding in either of $n = 2$ caves. George may announce that he will be searching forever and he will search Cave #1 on odd-numbered days and Cave #2 on even-numbered days. Osama the Second will hide out in Cave #2 on the first day, and move back and forth forever without getting caught.

Stupid people tend to be stubborn, but after a while, you convince George that his plan is not getting him anywhere. Reluctantly, he scraps his hare-brained scheme, and puts yours into action. He announces that he will be searching for two days and he will search Cave #2 on both days. If Osama the Second starts in Cave #2, he will be caught immediately. If not, he will come to Cave #2 the next day and be caught.

Do we need three days to catch Osama the Third, who is hiding in one of $n = 3$ caves? No, we can do better than that. George can announce that he will be searching for two days and will search Cave #2 on both days. After all, it worked last time.

Osama the Fourth is hiding in one of $n = 4$ caves. George announces that he will be searching for four days before he even figures out which cave he will search on each day. Then he tells you to finish his sentence. Can you wriggle out of this situation successfully?

Since we can search any one of four caves on each day, there are $4^4 = 256$ four-day plans. By symmetry, we may assume that we search either Cave #1 or Cave #2 on the first day. This cuts the number down to 128. The first plan, (1,1,1,1), is no good since Osama the Fourth can answer with (3,4,3,4). This is also the answer to a few other plans such as (4,2,1,2) and (2,3,1,1). In fact, you can verify that Osama the Fourth has an answer to each of the 128 plans.

Here is a five-day plan that works: (2,2,3,3,2). If Osama the Fourth starts in Cave #1, he will be caught on the second day. If he starts in Cave #2, he will be caught immediately. If he starts in Cave #3, he will either be caught on the second day if he moves to Cave #2, or on the fourth day in Cave #3. Finally, if he starts in Cave #4, he will go to Cave #3 on the second day. If he moves back to Cave #4 on the third day, he will be caught on the fourth day. If instead he moves to Cave #2 on the third day, he will be caught either on the fourth day in Cave #3 if he moves there, or on the fifth day in Cave #2 if he moves to Cave #1 instead.

Can you find a way of catching Osama for sure when $n \geq 5$? Of course, you should try to search for as few days as possible, since George claims that he regrets the loss of civilian lives while he is searching.