

# 49<sup>th</sup> International Mathematical Olympiad

Madrid, Spain

Day I

July 16, 2008

1. An acute-angled triangle  $ABC$  has orthocentre  $H$ . The circle passing through  $H$  with centre the midpoint of  $BC$  intersects the line  $BC$  at  $A_1$  and  $A_2$ . Similarly, the circle passing through  $H$  with centre the midpoint of  $CA$  intersects the line  $CA$  at  $B_1$  and  $B_2$ , and the circle passing through  $H$  with centre the midpoint of  $AB$  intersects the line  $AB$  at  $C_1$  and  $C_2$ . Show that  $A_1, A_2, B_1, B_2, C_1, C_2$  lie on a circle.
2. (a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers  $x, y, z$ , each different from 1, and satisfying  $xyz = 1$ .

- (b) Prove that equality holds above for infinitely many triples of rational numbers  $x, y, z$ , each different from 1, and satisfying  $xyz = 1$ .
3. Prove that there exist infinitely many positive integers  $n$  such that  $n^2 + 1$  has a prime divisor which is greater than  $2n + \sqrt{2n}$ .

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4. Find all functions  $f : (0, \infty) \rightarrow (0, \infty)$  (so,  $f$  is a function from the positive real numbers to the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers  $w, x, y, z$ , satisfying  $wx = yz$ .

5. Let  $n$  and  $k$  be positive integers with  $k \geq n$  and  $k - n$  an even number. Let  $2n$  lamps labelled  $1, 2, \dots, 2n$  be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

Let  $N$  be the number of such sequences consisting of  $k$  steps and resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off.

Let  $M$  be the number of such sequences consisting of  $k$  steps, resulting in the state where lamps  $1$  through  $n$  are all on, and lamps  $n + 1$  through  $2n$  are all off, but where none of the lamps  $n + 1$  through  $2n$  is ever switched on.

Determine the ratio  $N/M$ .

6. Let  $ABCD$  be a convex quadrilateral with  $|BA| \neq |BC|$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to the ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents of  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .