45^{th} International Mathematical Olympiad
Athens, Greece

Day I
July 12, 2004

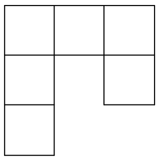
1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of the side BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

2. Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$

for all real numbers a, b, c such that $ab + bc + ca = 0$.

3. Define a *hook* to be a figure made up of six unit squares as shown in the diagram



or any of the figures obtained by applying rotations and reflections to this figure.

Determine all $m \times n$ rectangles that can be covered with hooks so that

- the rectangle is covered without gaps and without overlaps;
- no part of a hook covers area outside the rectangle.

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4. Let $n \geq 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

   Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

5. In a convex quadrilateral $ABCD$ the diagonal BD bisects neither the angle ABC nor the angle CDA. A point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA\,.$$

   Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

   Find all positive integers n such that n has a multiple which is alternating.