43^{rd} International Mathematical Olympiad
Glasgow, UK

Day I
July 24, 2002

1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is coloured red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$.

   Define an X-set to be a set of n blue points having distinct x-coordinates, and Y-set to be a set of n blue points having distinct y-coordinates.

   Prove that the number of X-sets is equal to the number of Y-sets.

2. Let BC be a diameter of the circle Γ with centre O. Let A be a point on Γ such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

   is an integer.

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Day II
July 25, 2002

4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$ where

$$1 = d_1 < d_2 < \cdots < d_k = n \, .$$

Define $D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$.

(a) Prove that $D < n^2$.

(b) Determine all n for which D is a divisor of n^2.

5. Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$(f(x) + f(z))\,(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all x, y, z, t in $\mathbb{R}$.

6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centres by $O_1, O_2, \ldots, O_n$ respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4} \, .$$