

37th International Mathematical Olympiad

Solutions

Problem 1

We shall work on the array \mathcal{A} of lattice points defined by

$$\mathcal{A} = \{(i, j) \in \mathbf{Z}^2 : 0 \leq i \leq 19, 0 \leq j \leq 11\}.$$

Our task is to move from $(0, 0)$ to $(19, 0)$ via the points of \mathcal{A} such that each move has length \sqrt{r} . Thus for each move of the form $(x, y) \rightarrow (x + a, y + b)$, we must have $a^2 + b^2 = r$.

(a) If r is even, then for each solution (a, b) of $a^2 + b^2 = r$, the sum $a + b$ is even, so for each lattice point (x, y) reached from $(0, 0)$, the parity of $x + y$ must be the same as that of $0 + 0$; that is, $x + y$ must be even. It follows that $(19, 0)$ cannot be reached from $(0, 0)$.

If r is a multiple of 3, then for each solution (a, b) of $a^2 + b^2 = r$, both a and b must be multiples of 3; this holds because -1 is not a square modulo 3. Thus for each lattice point (x, y) reached from $(0, 0)$, x and y must both be multiples of 3, and so in this case too $(19, 0)$ cannot be reached from $(0, 0)$.

(b) Consider the case $r = 73 = 8^2 + 3^2$. Let a, b, c and d represent, respectively, the number of moves of the types $\pm(8, 3)$, $\pm(8, -3)$, $\pm(3, 8)$ and $\pm(3, -8)$. (More precisely, a is the number of moves of type $(8, 3)$ minus the number of moves of type $(-8, -3)$; similarly for the others.) Since we have to reach $(19, 0)$ from $(0, 0)$, we have

$$8(a + b) + 3(c + d) = 19, \quad 3(a - b) + 8(c - d) = 0.$$

Taking $(a + b, c + d) = (2, 1)$ as a trial solution of the first equation, and $(a - b, c - d) = (-8, 3)$ as a trial solution of the second, we find that

$$a = -3, b = 5, c = 2, d = -1.$$

We now attempt a solution with three moves of type $(-8, -3)$, five moves of type $(8, -3)$, two moves of type $(3, 8)$ and one of type $(-3, 8)$. The constraint is that we must keep within the boundaries of the array. After some experimentation, the following route emerges:

$$(0, 0) \rightarrow (8, 3) \rightarrow (11, 5) \rightarrow (19, 2) \rightarrow (16, 10) \rightarrow (8, 7) \rightarrow (0, 4) \rightarrow (8, 1) \rightarrow (11, 9) \rightarrow (3, 6) \rightarrow (11, 3) \rightarrow (19, 0).$$

Note that the solution $(a + b, c + d) = (2, 1)$, $(a - b, c - d) = (8, -3)$, which gives $a = 5, b = 3, c = 1$ and $d = 2$, also yields a route:

$(0,0) \rightarrow (8,3) \rightarrow (16,6) \rightarrow (8,9) \rightarrow (5,1) \rightarrow (13,4) \rightarrow (5,7) \rightarrow (13,10) \rightarrow (16,2) \rightarrow (8,5) \rightarrow (16,8) \rightarrow (19,0)$.

(c) If $r = 97$, then since the only way of writing 97 as the sum of two squares is $97 = 9^2 + 4^2$, each of the moves must consist of one of the vectors $(\pm 9, \pm 4)$, $(\pm 4, \pm 9)$. Let the points of \mathcal{A} be partitioned as $\mathcal{B} \cup \mathcal{C}$ in the following manner:

$$\mathcal{B} = \{(i, j) \in \mathbf{Z}^2 : 0 \leq i \leq 19, 4 \leq j \leq 7\}, \quad \mathcal{C} = \mathcal{A} \setminus \mathcal{B}.$$

Then it can be verified that moves of the type $(\pm 9, \pm 4)$ always take us from points in \mathcal{B} to points in \mathcal{C} and *vice versa*, while moves of type $(\pm 4, \pm 9)$ always take us from points in \mathcal{C} to points in \mathcal{B} . (Note that it is not possible to go from one point in \mathcal{B} to another point in \mathcal{B} in one step.)

Each move of the type $(\pm 9, \pm 4)$ changes the parity of the x -coordinate, so since we have to go from $(0,0)$ to $(19,0)$, and *odd* number of such moves is required. Each such move takes us from \mathcal{B} to \mathcal{C} or *vice versa*, so since the starting point $(0,0)$ is in \mathcal{C} , we shall end up at a point in \mathcal{B} . However, $(19,0) \in \mathcal{C}$. It follows that the required sequence of moves does not exist.

Problem 2

Lemma: Let the feet of the perpendiculars from P to BC , CA and AB be X , Y and Z respectively. Then (i) $YZ = PA \sin A$ (ii) $\angle YXZ = \angle BPC - \angle A$.

This is easy to see via an examination of the three cyclic quadrilaterals $AZPY$, $BXPZ$ and $CYPX$.

Let BD and CE meet AP in Q and R respectively. By the angle bisector theorem, $AQ/QP = AB/BP$ and $AR/RC = AC/CP$. To show that Q, R coincide, it suffices to show that $AB/BP = AC/CP$. Now,

$$\begin{aligned} \frac{AB}{BP} = \frac{AC}{CP} &\iff AB \cdot CP = AC \cdot BP \iff CP \cdot \sin C = BP \cdot \sin B \\ &\iff XY = XZ \quad (\text{using the Lemma}). \end{aligned}$$

But we are given that $\angle APB - \angle C = \angle APC - \angle B$. This implies that $\angle XZY = \angle XYZ$ (also by the Lemma), so $XY = XZ$ as desired.

Problem 3

Putting $m = n = 0$ we obtain $f(0) = 0$ and hence $f(f(n)) = f(n)$ for all $n \in \mathbf{N}_0$. Thus the given functional equation is equivalent to

$$f(m + f(n)) = f(m) + f(n), \quad f(0) = 0.$$

We also observe that if $f(x)$ is not the zero function then f has non-zero fixed points. Let a be the least non-zero fixed point of f . If $a = 1$ then it is easy to check that $f(2) = 2$ and by induction that $f(n) = n$ for all $n \in \mathbf{N}_0$.

Suppose $a > 0$. Again by induction $f(ka) = ka$ for all $k \geq 1$. We shall show that the fixed points of f are all of the form ka for some $k \geq 1$. First note that the sum of two fixed points of f is itself a fixed point. Let b be an arbitrary fixed point of f . Choose non-negative integers q, r such that $b = aq + r, 0 \leq r < a$. Then we get

$$b = f(b) = f(aq + r) = f(r + f(aq)) = f(r) + f(aq) = f(r) + aq.$$

It follows that $f(r) = r$ and since $r < a$ we must have $r = 0$. This proves the claim that the fixed points are all of the form ka . Since the set $\{f(n) : n \in \mathbf{N}_0\}$ is a set of fixed points of f it follows in particular that $f(i) = an_i$ for each $i < a$, with $n_0 = 0$ and $n_i \in \mathbf{N}_0$.

Take any positive integer n and write it as $n = ka + i$ where $0 \leq i < a$. Using the functional equation we obtain

$$f(n) = f(i + ka) = f(i + f(ka)) = f(i) + ka = n_i a + ka = (n_i + k)a.$$

We verify that such an f satisfies the given functional equation: take $m = ka + i, n = la + j, 0 \leq i, j < a$. Then

$$\begin{aligned} f(m + f(n)) &= f(ka + i + f(la + j)) = f((k + l + n_j)a + i) \\ &= (k + l + n_j + n_i)a \\ &= f(m) + f(n) \end{aligned}$$

Thus if f is not identically zero, then f has the following general form: let $a \in \mathbf{N}$ and $n_1, n_2, \dots, n_{a-1} \in \mathbf{N}_0$ be chosen arbitrarily; then

$$f(n) = \left(\left\lfloor \frac{n}{a} \right\rfloor + n_i \right) a.$$

Problem 4

Let $15a + 16b = r^2, 16a - 15b = s^2$, where $r, s \in \mathbf{N}$. We now obtain:

$$r^4 + s^4 = (15^2 + 16^2)(a^2 + b^2) = 481(a^2 + b^2).$$

Note that $481 = 13 \times 37$. We now use the fact that -1 is not a fourth power either modulo 13 or modulo 37. (To see why this holds, note that the congruence $-1 \equiv x^4 \pmod{13}$ for some $x \in \mathbf{N}$ leads via Fermat's theorem, to $(-1)^3 \equiv 1 \pmod{13}$,

which is false; likewise, the congruence $-1 \equiv x^4 \pmod{37}$ for some $x \in \mathbf{N}$ leads to $(-1)^9 \equiv 1 \pmod{37}$, which too is false.)

Since $r^4 + s^4 \equiv 0 \pmod{13}$, either $r \equiv s \equiv 0 \pmod{13}$ or $r \not\equiv 0, s \not\equiv 0$ (both modulo 13). The latter possibility cannot occur because -1 is not a fourth power modulo 13; therefore $r \equiv s \equiv 0 \pmod{13}$, and similarly $r \equiv s \equiv 0 \pmod{37}$. Therefore r and s are both multiples of 481, and so $r \geq 481, s \geq 481$. It is easy to check that $r = s = 481$ is realizable: we obtain

$$a = 481 \cdot 31, \quad b = 481.$$

Thus the required answer is 481^2 .

Problem 5

Let a, b, c, d, e and f denote the lengths of the sides AB, BC, DE, EF and FA respectively. Note that the opposite angles of the hexagon are equal ($\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$). Draw perpendiculars as follows: $AP \perp BC, AS \perp EF, DQ \perp EF$. Then $PQRS$ is a rectangle and $BF \geq PS = QR$. Therefore $2BF \geq PS + QR$, and so

$$2BF \geq (a \sin B + f \sin C) + (c \sin C + d \sin B).$$

Similarly,

$$2DB \geq (c \sin A + b \sin B) + (e \sin B + f \sin A),$$

$$2FD \geq (e \sin C + d \sin A) + (a \sin A + b \sin C).$$

Next, the circumradii of the triangles FAB, BCD and DEF are related to BF, DB and FD as follows:

$$R_A = \frac{BF}{2 \sin A}, \quad R_C = \frac{DB}{2 \sin C}, \quad R_E = \frac{FD}{2 \sin B}.$$

We obtain, therefore,

$$4(R_A + R_C + R_E) \geq a \left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B} \right) + b \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} \right) + \dots$$

$$\geq 2(a + b + \dots) = 2P,$$

and so $R_A + R_B + R_C \geq P/2$, as required. Equality holds iff $\angle A = \angle B = \angle C$ and $BF \perp BC, \dots$; that is, iff the hexagon is regular.

Problem 6

We first remark that there is no loss in taking p and q to be coprime; for, if p, q have a common factor $d > 1$, then we can reword the problem in terms of the quantities $p' = p/d$, $q' = q/d$, $x'_i = x_i/d$.

Let there be k indices $i \in \{1, 2, \dots, n\}$ such that $x_i - x_{i-1} = p$; then the number of indices $i \in \{1, 2, \dots, n\}$ such that $x_i - x_{i-1} = -q$ is $n - k$. Since $x_n = x_0 = 0$, we see that $kp = (n - k)q$, and since p, q are coprime this implies that $k = aq$, $n - k = ap$ for some positive integer a . It follows that $n = a(p + q)$, and since $n > p + q$, we have $a > 1$.

Let $y_i = x_{i+p+q} - x_i$ for $i \in \{0, 1, \dots, n - p - q\}$. Since $n > p + q$, there is more than one y_i . We shall show that at least one of the y_i is 0, which will establish the stated claim. (In fact, this establishes a stronger statement.)

For each i , let S_i denote the set of indices $\{i + 1, i + 2, \dots, i + p + q\}$. Let r be the number of $j \in S_i$ for which $x_j - x_{j-1} = p$; then the number of $j \in S_i$ for which $x_j - x_{j-1} = -q$ is $p + q - r$. Summing these equalities over all $j \in S_i$, we obtain

$$y_i = rp - (p + q - r)q = (p + q)(r - q).$$

Thus y_i is a multiple of $(p + q)$ for each i . Now consider the expression $y_{i+1} - y_i$:

$$\begin{aligned} y_{i+1} - y_i &= (x_{i+p+q+1} - x_{i+1}) - (x_{i+p+q} - x_i) \\ &= (x_{i+p+q+1} - x_{i+p+q}) - (x_{i+1} - x_i) \end{aligned}$$

Since each bracketed term is p or $-q$, it follows that $y_{i+1} - y_i$ is 0 or $pm(p + q)$. Next, consider the relation:

$$y_0 + y_{p+q} + y_{2(p+q)} + \dots + y_{n-p-q} = 0.$$

This shows that the y_i 's are neither all positive or all negative. Thus in the sequence

$$y_0, y_1, y_2, \dots, y_{n-p-q-1}, y_{n-p-q},$$

there exists two adjacent y 's that are not of the same sign. Since each y_i is a multiple of $(p + q)$, and since the difference between adjacent y_i 's is always 0 or $\pm(p + q)$, it follows that some y_i equals 0.