THE 36^{th} IMO, CANADA, 1995

1 First Day, July 19, 1995

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN, and XY are concurrent.

2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ such that there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

   (a) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;

   (b) for each triple $i, j, k (1 \leq i < j < k \leq n)$ the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

2 Second Day, July 20, 1995

1. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

   (a) $x_0 = x_{1995}$;

   (b) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, \ldots, 1995.$

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = 60°.$

   Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°.$ Prove that $AG + GB + GH + DH + HE \geq CF.$

3. Let p be an odd prime. Find the number of subsets A of $\{1, 2, \ldots, 2p\}$ such that

   (a) A has exactly p elements, and

   (b) the sum of all the elements in A is divisible by $p.$

SOLUTIONS

3 First Day, July 19, 1995

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN, and XY are concurrent.

Solution

Let the lines XY and DN meet at the point Q. Then triangles BPZ and QDZ are similar. Hence $\frac{ZQ}{ZD} = \frac{ZB}{ZP}$ or $ZQ = \frac{ZD \cdot ZB}{ZP}$. Since Z, D, B and P are fixed points, so is Q. By symmetry, AM also passes through Q.

Alternative Solution

Draw a line through A parallel to BP, cutting the line XY at the point $W.$ Then triangles BPZ and AWZ are similar. Hence $\frac{ZP}{ZW} = \frac{ZB}{ZA}.$ Since

$$ZA \cdot ZC = ZX \cdot ZY = ZB \cdot ZD,$$

we have $\frac{ZP}{ZW} = \frac{ZC}{ZD}.$ It follows that triangles CPZ and DWZ are also similar, so that DW is parallel to $CP.$ Since AM is perpendicular to $CP,$ it is also perpendicular to $DW.$ Similarly, DN is perpendicular to $AW.$ Finally, XY is perpendicular to AD. Hence AM, DN and XY are concurrent at the orthocentre of triangle $WAD.$

Remark: This problem is very easy, and has many different solutions. They may be grouped into two types. The first solution, due independently to Sam Maltby and Johannes Notenboom, the leader of the team from the Netherlands, represents those in which only half of the diagram is considered. The second solution, due to Nazar Agakhanov, the leader of the team from Russia, represents those in which the whole diagram is considered. Other techniques used include Menelaus' Theorem and analytic geometry.

2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \;\geq\; \frac{3}{2}\,.$$

Solution

Let

$$S = \frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}$$

and define

$$x = \frac{1}{a}, \quad y = \frac{1}{b} \quad z = \frac{1}{c} \quad \text{and} \quad T = x + y + z.$$

The positive real numbers $x, y,$ and z satisfy $xyz = 1.$ Then

$$\frac{1}{a^3(b+c)} = \frac{x^3}{\frac{1}{y} + \frac{1}{z}} = \frac{x^3yz}{y+z} = \frac{x^3yz}{y+z} = \frac{x^2}{T-x}$$

$$= \frac{T^2 - (T^2 - x^2)}{T-x} = \frac{T^2}{T-x} - T - x.$$

This and the corresponding terms in y and z yield

$$T^2\left(\frac{1}{T-x} + \frac{1}{T-y} + \frac{1}{T-z}\right) - 4T.$$

By the Arithmetic–Harmonic-Mean Inequality

$$S \geq T^2 \cdot \frac{9}{(T-x)+(T-y)+(T-z)} - 4T = \frac{9T}{2} - 4T = \frac{x+y+z}{2}.$$

By the Arithmetic–Geometric-Mean Inequality

$$S \geq \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}.$$

Equality holds if and only if $x = y = z = 1,$ which is equivalent to $a = b = c = 1.$

Alternative Solution

Let x, y, z and S be as in the first solution.

$$S = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

By Cauchy's Inequality,

$$[(y+z)+(z+x)+(x+y)]\,S \geq (x+y+z)^2$$

or $S \geq \frac{x+y+z}{2}$. It follows from the Arithmetic–Geometric-Mean Inequality that

$$S \geq \frac{x+y+z}{3} \cdot \frac{3}{2} \geq \sqrt[3]{xyz} \cdot \frac{3}{2} = \frac{3}{2}.$$

Equality holds if and only if $x = y = z = 1$, which is equivalent to $a = b = c = 1$.

Remark: The first solution is due to the proposer, Nazar Agakhanov, the leader of the team from Russia. The second solution is due to Murray Klamkin. Several competitors applied Chebychev's Inequality directly to S (written in terms of x, y and z) followed by applications of the Root-Mean-Square–Arithmetic-Mean Inequality and Arithmetic–Geometric-Mean Inequality.

3. Determine all integers $n > 3$ such that there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

(a) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;

(b) for each triple $i, j, k (1 \leq i < j < k \leq n)$ the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Solution

We claim that $n = 4$ is the only integer satisfying the conditions of the problem. For $n = 4$, let $A_1A_2A_3A_4$ be a unit square and let $r_1 = r_2 = r_3 = r_4 = 1/6$. It remains to show that no solution exists for $n = 5$, which implies that there are no solutions for any $n \geq 5$.

Suppose to the contrary that there is a solution with $n = 5$. Denote the area of $\triangle A_iA_jA_k$ by $[ijk] = r_i + r_j + r_k, 1 \leq i < j < k \leq 5$. If $A_iA_jA_kA_\ell$ is convex, then $r_i + r_k = r_j + r_\ell$. This follows from $[ijk] + [k\ell i] = [jk\ell] + [\ell ij]$.

We cannot have $r_i = r_j$. If for instance $r_4 = r_5$, then $[124] = [125]$. If A_1 and A_2 are on the same side of A_4A_5, then A_1A_2 must be parallel to A_4A_5. If they are on opposite sides, then A_1A_2 must pass through the midpoint M of A_4A_5. The same can be said about A_2A_3 and A_3A_1. Since A_1, A_2 and A_3 are not collinear, at most one of A_1A_2, A_2A_3 and A_3A_1 can be parallel to A_4A_5, and at most one can pass through M. This is a contradiction.

Consider the convex hull of A_1, A_2, A_3, A_4 and A_5. We have three cases.

First, suppose that the convex hull is a pentagon $A_1A_2A_3A_4A_5$. Since $A_1A_2A_3A_4$ and $A_1A_2A_3A_5$ are convex, our observation yields $r_1+r_3 = r_2+r_4$ and $r_1+r_3 = r_2+r_5$. Hence $r_4 = r_5$, a contradiction.

Next, suppose that the convex hull is a quadrilateral $A_1A_2A_3A_4$. We may assume that A_5 lies within $A_3A_4A_1$. Then $A_1A_2A_3A_5$ is convex, and we have the same contradiction as before.

Finally, suppose that the convex hull is a triangle $A_1A_2A_3$. Since

$$[124] + [234] + [314] = [125] + [235] + [315],$$

we have $r_4 = r_5$, a contradiction.

Alternative Solution

We proceed as in the first solution up to $r_i + r_k = r_j + r_\ell$ if $A_iA_jA_kA_\ell$ is convex. Assume that $r_1 \geq r_2 \geq r_3 \geq r_4 \geq r_5$. Then [123] is the greatest area among the ten triangles determined by these five points. Hence both A_4 and A_5 lie within the triangle $B_1B_2B_3$ which has A_1, A_2 and A_3 as the midpoints of its sides B_2B_3, B_3B_1 and B_1B_2, respectively.

Suppose both A_4 and A_5 are inside $A_1A_2A_3$. Then

$$[124] + [234] + [314] = [125] + [235] + [315],$$

which implies that $r_4 = r_5$. Since A_4 and A_5 are on the same side of A_1A_2 and of A_2A_3, A_4A_5 must be parallel to both segments. This is a contradiction since A_1, A_2 and A_3 are not collinear.

We may now assume by symmetry that A_4 is in $B_1A_2A_3$. Then $r_1+r_4 = r_2+r_3$. If A_5 is also in $B_1A_2A_3$, then $r_1+r_5 = r_2+r_3$ so that $r_4 = r_5$. This leads to a contradiction as before. On the other hand, if A_5 is in $A_2A_3B_2B_3$, then $r_4 + r_5 = r_2 + r_3$ so that $r_1 = r_5$. This also leads to a contradiction.

Remark: The first solution is due to Bill Sands. The second solution is due to the proposer of the problem, Karel Horak, leader of the team from the Czech Republic.

4 Second Day, July 20, 1995

1. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

   (a) $x_0 = x_{1995}$;

   (b) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, \ldots, 1995.$

   Solution

   The given condition is equivalent to $x_i^2 - \left(\frac{x_{i-1}}{2} + \frac{1}{x_{i-1}}\right) x_i + \frac{1}{2} = 0,$ which yields either $x_i = \frac{1}{2}x_{i-1}$ or $x_i = \frac{1}{x_{i-1}}.$

   We call the transition from x_{i-1} to x_i a move. Starting from $x_0,$ all possible moves are represented by arrows in Figure 1. The halving moves are represented by solid lines, while the reciprocating moves are represented by broken lines. We have to return to x_0 after exactly 1995 moves.

   Note that each row consists of distinct numbers as long as $x_0 \neq 0,$ and the two rows of numbers are either disjoint or identical. If they are disjoint, then it is not possible to return to x_0 after an odd number of moves, since each move is between a number enclosed by a circle and one by a square.

   It follows that the two rows of numbers are identical. Even in this case, the task is only possible if the numbers in the first row enclosed by circles are identical to those in the second enclosed by squares. Moreover, the numbers in the two rows are descending in opposite direction. It follows that one of the numbers must be equal to 1, so that we can replace Figure 1 by Figure 2.

   In order to maximize $x_0,$ there is no point in taking a reciprocal except on the very last move. Of the remaining 1994 moves, exactly half will be made on each side of 1. Hence the maximum value of x_0 is $2^{997}.$

Alternative Solution

As in the first solution, either $x_i = \frac{x_{i-1}}{2}$ or $x_i = \frac{1}{x_{i-1}}$. For $i \geq 0$, we claim that $x_i = 2^{k_i} x_0^{\epsilon_i}$ for some integer k_i with $|k_i| \leq i$ and $\epsilon_i = (-1)^{k_i+i}$. This is true for $i = 0$, with $k_0 = 0$ and $\epsilon_0 = 1$, and we proceed by induction. If it is true for $i-1$ and $x_i = \frac{1}{2}x_{i-1}$, then we have $k_i = k_{i-1} - 1$ and $\epsilon_i = \epsilon_{i-1}$; while if $x_i = \frac{1}{x_{i-1}}$, then we have $k_i = -k_{i-1}$ and $\epsilon_i = -\epsilon_{i-1}$. In each case, it is immediate that $|k_i| \leq i$ and $\epsilon_i = (-1)^{k_i+i}$. Thus $x_{1995} = 2^k x_0^{\epsilon}$, where $k = k_{1995}$ and $\epsilon = \epsilon_{1995}$, with $0 \leq |k| \leq 1995$ and $\epsilon = (-1)^{1995+k}$. It follows that $x_0 = x_{1995} = 2^k x_0^{\epsilon}$. If k is odd, then $\epsilon = 1$ and we have $2^k = 1$, a contradiction since $k \neq 0$. Thus k must be even, so that $\epsilon = -1$ and $x_0^2 = 2^k$. Since k is even and $|k| \leq 1995$, $k \leq 1994$. Hence $x_0 \leq 2^{997}$. We can have $x_0 = 2^{997}, x_i = \frac{1}{2}x_{i-1}$ for $i = 1, 2, \cdots, 1994$, and $x_{1995} = \frac{1}{x_{1994}}$. Then

$$x_{1995} = \frac{1}{2^{-1994}x_0} = x_0$$

as desired.

Remark: The first solution is due to Johannes Notenboom, the leader of the team from the Netherlands. It is along the line of that of the proposer of the problem, Marcin Kuczma, the leader of the team from Poland, differing only in presentation. The second solution is due to Sam Maltby.

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = 60^\circ$.

   Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^\circ$. Prove that $AG + GB + GH + DH + HE \geq CF$.

   Solution

   Note that BCD and EFA are equilateral triangles. It follows that BE is an axis of symmetry of $ABDE$. Reflect BCD and EFA about BE to $BC'A$ and $EF'D$ respectively. Since $\angle BGA = 180^\circ - \angle AC'B$, G lies on the circumcircle of ABC'. Hence $\angle AGC' = \angle ABC' = 60^\circ$. Let K be the point on GC' such that KAG is equilateral. Then $\angle C'AK = 60^\circ - \angle BAK = \angle BAG$. Since $C'A = BA$ and $AK = AG$, triangles $C'AK$ and BAG are congruent. It follows that $GC' = GK + KC' = GA + GB$. Similarly, $DH + HE = HF'$. Hence

   $$CF = C'F' \leq C'G + GH + HF' = AG + GB + GH + DH + HE,$$

   with equality if and only if C', G, H and F' are collinear in that order.

Alternative Solution

The result holds without the condition that $\angle AGB = \angle DHE = 120°.$ Let C' and F' be as in the first solution. By Ptolemy's Inequality, $GC' \cdot AB \geq GA \cdot BC' + GB \cdot AC'$ so that $GC' \geq GA + GB.$ Similarly, $HF' \geq HD + HE,$ It follows that

$$CF = C'F' \leq C'G + GH + HF' \leq AG + GB + GH + HD + HE.$$

Remark: The first solution is due to Bill Sands who observes that the lemma $GC' = GA + GB$ is a special case of Ptolemy's Theorem which is featured in a 1973 Putnam Mathematics Competition. The second solution is due to Arthur Baragar. He uses Ptolemy's Inequality which is a stronger result than Ptolemy's Theorem.

3. Let p be an odd prime. Find the number of subsets A of $\{1, 2, \ldots, 2p\}$ such that

(a) A has exactly p elements, and

(b) the sum of all the elements in A is divisible by p.

Solution

For any p-element subset A of $\{1, 2, ..., 2p\}$, denote by $s(A)$ the sum of the elements of A. Of the $\binom{2p}{p}$ such subsets, $B = \{1, 2, ..., p\}$ and $C = \{p+1, p+2, ..., 2p\}$ satisfy $s(B) = s(C) \equiv 0 \pmod{p}$. For $A \neq B, C$, we have $A \bigcap B \neq \emptyset \neq A \bigcap C$. Partition the $\binom{2p}{p} - 2$ p-element subsets other than B and C into groups of size p as follows. Two subsets A and A' are in the same group if and only if $A' \bigcap C = A \bigcap C$ and $A' \bigcap B$ is a cyclic permutation of $A \bigcap B$ within B. Suppose $A \bigcap B$ has n elements, $0 < n < p$. For some m such that $0 < m < p$,

$$A' \bigcap B = \{x + m : x \in A \bigcap B, x + m \leq p\}$$

$$\bigcup \{x + m - p : x \in A \bigcap B, x \leq p < x + m\}.$$

Hence $s(A') - s(A) \equiv mn \pmod{p}$, but mn is not divisible by p. It follows that exactly one subset A in each group satisfies $s(A) \equiv 0 \pmod{p}$, and the total number of such subsets is $p^{-1}\left(\binom{2p}{p} - 2\right) + 2$.

Alternative Solution

Let ω be a primitive p-th root of unity. Then

$$\prod_{i=1}^{2p}(x-\omega^i)=(x^p-1)^2=x^{2p}-2x^p+1.$$

Comparing the coefficients of the term x^p, we have

$$2=\sum \omega^{i_1+i_2+\cdots+i_p}=\sum_{j=0}^{p-1} n_j\omega^j,$$

where the first summation ranges over all subsets $\{i_1, i_2, \ldots, i_p\}$ of $\{1, 2, \ldots, 2p\}$ and n_j in the second summation is the number of such subsets such that $i_1+i_2+\cdots+i_n \equiv j \pmod p$. It follows that ω is a root of $G(x)=(n_0-2)+\sum_{j=1}^{p-1} n_j\omega^j$, which is a polynomial of degree $p-1$. Since the minimal polynomial for ω over the field of rational numbers is $F(x)=\sum_{j=0}^{p-1}\omega^j$, which is also of degree $p-1$, $G(x)$ must be a scalar multiple of $F(x)$, so that $n_0-2=n_1=n_2=\cdots=n_{p-1}$. Since $\sum_{j=0}^{p} n_j=\binom{2p}{p}$, we have $n_0=p^{-1}\left(\binom{2p}{p}-2\right)+2$.

Remark: The first solution is due to the proposer, Marcin Kuczma, the leader of the team from Poland. The second solution is due to Roberto Dvornicich, the leader of the team from Italy. Nikolay Nikolov, a Bulgarian student, won a special prize for his solution which is essentially along the line of the second one. Nikolay had won two Gold Medals and one Silver Medal at the last three IMO's, and topped off his outstanding career as a competitor by obtaining a perfect score this time.