SHORTLISTED PROBLEMS
for the 36^{th} IMO
CANADA, 1995

1 INTRODUCTION

The Problem Selection Committee of the 36^{th} International Mathematical Olympiad presents 28 problems for consideration by the Jury. They are classified under Algebra, Geometry, Number Theory & Combinatorics and Sequences. Within each group, they are arranged in ascending order of estimated difficulty.

A1	Russia	G1	Bulgaria	N1	Romania	S1	Ukraine
A2	Sweden	G2	Germany	N2	Russia	S2	Poland
A3	Ukraine	G3	Turkey	N3	Czech Republic	S3	Poland
A4	United States	G4	Ukraine	N4	Bulgaria	S4	New Zealand
A5	Ukraine	G5	New Zealand	N5	Ireland	S5	Finland
A6	Japan	G6	United States	N6	Poland	S6	India
		G7	Latvia	N7	Belarus		
		G8	Columbia	N8	Germany		

The following countries also contributed problems: Australia, Cyprus, Estonia, France, Great Britain, Hong Kong, Iran, Kazakhstan, Luxembourg, Macedonia, Mongolia, The Netherlands, Norway, South Korea, Spain, Taiwan, Thailand and Vietnam.

2 PROBLEMS

Algebra

1. Let a, b and c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}\,.$$

2. Let a and b be non-negative integers such that $ab \geq c^2$, where c is an integer. Prove that there is a number n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that
$$\sum_{i=1}^{n} x_i^2 = a, \quad \sum_{i=1}^{n} y_i^2 = b, \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i = c.$$

3. Let n be an integer, $n \geq 3$. Let $a_1, a_2, \ldots, a_n$ be real numbers, where $2 \leq a_i \leq 3$ for $i = 1, 2, \ldots, n$. If $s = a_1 + a_2 + \cdots + a_n$, prove that
$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \cdots + \frac{a_n^2 + a_1^2 - a_2^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

4. Let a, b and c be given positive real numbers. Determine all positive real numbers x, y and z such that
$$x + y + z = a + b + c$$
and
$$4xyz - (a^2x + b^2y + c^2z) = abc.$$

5. Let $\mathbb{R}$ be the set of real numbers. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ which simultaneously satisfies the following three conditions?

   (a) There is a positive number M such that $-M \leq f(x) \leq M$ for all x.

   (b) The value of $f(1)$ is 1.

   (c) If $x \neq 0$, then
$$f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2.$$

6. Let n be an integer, $n \geq 3$. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i < x_{i+1}$ for $1 \leq i \leq n-1$. Prove that
$$\frac{n(n-1)}{2} \sum_{i<j} x_i x_j > \left(\sum_{i=1}^{n-1} (n-i)x_i\right)\left(\sum_{j=2}^{n} (j-1)x_j\right).$$

Geometry

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

2. Let A, B and C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$.

3. The incircle of ABC touches BC, CA and AB at D, E and F respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

4. An acute triangle ABC is given. Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A) and C_1 and C_2 on the side AB (with C_2 between C_1 and B) so that

$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2$$
$$= \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$

The lines AA_1, BB_1 and CC_1 bound a triangle, and the lines AA_2, BB_2 and CC_2 bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

5. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = 60°$. Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Prove that $AG + GB + GH + DH + HE \geq CF$.

6. Let $A_1A_2A_3A_4$ be a tetrahedron, G its centroid, and A'_1, A'_2, A'_3 and A'_4 the points where the circumsphere of $A_1A_2A_3A_4$ intersects GA_1, GA_2, GA_3 and GA_4 respectively. Prove that

$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA_4 \leq GA'_1 \cdot GA'_2 \cdot GA'_3 \cdot GA'_4$$

and

$$\frac{1}{GA'_1} + \frac{1}{GA'_2} + \frac{1}{GA'_3} + \frac{1}{GA'_4} \leq \frac{1}{GA_1} + \frac{1}{GA_2} + \frac{1}{GA_3} + \frac{1}{GA_4}.$$

7. O is a point inside a convex quadrilateral $ABCD$ of area S. K, L, M and N are interior points of the sides AB, BC, CD and DA respectively. If $OKBL$ and $OMDN$ are parallelograms, prove that $\sqrt{S} \geq \sqrt{S_1} + \sqrt{S_2}$, where S_1 and S_2 are the areas of $ONAK$ and $OLCM$ respectively.

8. Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC' and HH' are concurrent, where H and H' are the orthocentres of triangles ABC and $AB'C'$ respectively.

Number Theory & Combinatorics

1. Let k be a positive integer. Prove that there are infinitely many perfect squares of the form $n2^k - 7$, where n is a positive integer.

2. Let $\mathbb{Z}$ denote the set of all integers. Prove that, for any integers A and B, one can find an integer C for which $M_1 = \{x^2 + Ax + B : x \in \mathbb{Z}\}$ and $M_2 = \{2x^2 + 2x + C : x \in \mathbb{Z}\}$ do not intersect.

3. Determine all integers $n > 3$ such that there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

   (a) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;

   (b) for each triple $i, j, k (1 \leq i < j < k \leq n)$, the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

4. Find all positive integers x and y such that $x + y^2 + z^3 = xyz$, where z is the greatest common divisor of x and y.

5. At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

6. Let p be an odd prime number. Find the number of subsets A of $\{1, 2, \ldots, 2p\}$ such that

   (a) A has exactly p elements, and

   (b) the sum of all the elements in A is divisible by p.

7. Does there exist an integer $n > 1$ which satisfies the following condition?

   The set of positive integers can be partitioned into n non-empty subsets, such that an arbitrary sum of $n-1$ integers, one taken from each of any $n-1$ of the subsets, lies in the remaining subset.

8. Let p be an odd prime. Determine positive integers x and y for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is non-negative and as small as possible.

Sequences

1. Does there exist a sequence $F(1),\ F(2),\ F(3),\ \ldots$ of non-negative integers which simultaneously satisfies the following three conditions?

   (a) Each of the integers $0, 1, 2, \ldots$ occurs in the sequence.

(b) Each positive integer occurs in the sequence infinitely often.

(c) For any $n \geq 2$,

$$F(F(n^{163})) = F(F(n)) + F(F(361)).$$

2. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

(a) $x_0 = x_{1995}$;

(b) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, 2, \ldots, 1995$.

3. For an integer $x \geq 1$, let $p(x)$ be the least prime that does not divide x, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$. For x having $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n \geq 0$. Find all n such that $x_n = 1995$.

4. Suppose that $x_1, x_2, x_3, \ldots$ are positive real numbers for which

$$x_n^n = \sum_{j=0}^{n-1} x_n^j$$

for $n = 1, 2, 3, \ldots$. Prove that for all n,

$$2 - \frac{1}{2^{n-1}} \leq x_n < 2 - \frac{1}{2^n}.$$

5. For positive integers n, the numbers $f(n)$ are defined inductively as follows: $f(1) = 1$, and for every positive integer n, $f(n+1)$ is the greatest integer m such that there is an arithmetic progression of positive integers $a_1 < a_2 < \cdots < a_m = n$ and

$$f(a_1) = f(a_2) = \cdots = f(a_m).$$

Prove that there are positive integers a and b such that $f(an+b) = n+2$ for every positive integer n.

6. Let $\mathbb{N}$ denote the set of all positive integers. Prove that there exists a unique function $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(m + f(n)) = n + f(m + 95)$$

for all m and n in $\mathbb{N}$. What is the value of $\sum_{k=1}^{19} f(k)$?

3 SOLUTIONS

3.1 ALGEBRA

1. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution

Let

$$S = \frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}$$

and define

$$x = \frac{1}{a},\ y = \frac{1}{b},\ z = \frac{1}{c} \text{ and } T = x + y + z.$$

The positive real numbers $x, y,$ and z satisfy $xyz = 1$. Then

$$\begin{aligned}
\frac{1}{a^3(b+c)} &= \frac{x^3}{\frac{1}{y} + \frac{1}{z}} \\
&= \frac{x^3yz}{y+z} = \frac{x^3yz}{y+z} \\
&= \frac{x^2}{T-x} = \frac{T^2 - (T^2 - x^2)}{T-x} \\
&= \frac{T^2}{T-x} - T - x.
\end{aligned}$$

This and the corresponding terms in y and z yield

$$S = T^2\left(\frac{1}{T-x} + \frac{1}{T-y} + \frac{1}{T-z}\right) - 4T.$$

By the Arithmetic–Harmonic-Mean Inequality

$$S \geq T^2 \cdot \frac{9}{(T-x)+(T-y)+(T-z)} - 4T = \frac{9T}{2} - 4T = \frac{x+y+z}{2}.$$

By the Arithmetic–Geometric-Mean Inequality

$$S \geq \frac{3}{2}\sqrt[3]{xyz} = \frac{3}{2}.$$

Equality holds if and only if $x = y = z = 1$, which is equivalent to $a = b = c = 1$.

Alternative Solution

Let x, y, z and S be as in the first solution.

$$S = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

By Cauchy's Inequality,

$$[(y+z)+(z+x)+(x+y)]\, S \geq (x+y+z)^2$$

or $S \geq \frac{x+y+z}{2}$. It follows from the Arithmetic–Geometric-Mean Inequality that

$$S \geq \frac{x+y+z}{3} \cdot \frac{3}{2} \geq \sqrt[3]{xyz} \cdot \frac{3}{2} = \frac{3}{2}.$$

Equality holds if and only if $x = y = z = 1$, which is equivalent to $a = b = c = 1$.

Remark: This is Problem 2 of the 36th IMO on July 19, 1995. The first solution is due to the proposer, Nazar Agakhanov, the leader of the team from Russia. The second solution is due to Murray Klamkin of our committee. Several competitors applied Chebychev's Inequality directly to S (written in terms of x, y and z) followed by applications of the Root-Mean-Square–Arithmetic-Mean Inequality and Arithmetic–Geometric-Mean Inequality.

2. Let a and b be non-negative integers such that $ab \geq c^2$, where c is an integer. Prove that there is a number n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \quad \sum_{i=1}^{n} y_i^2 = b, \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i = c.$$

Solution

We note that the statement is true for (a, b, c) if and only if it is true for $(a, b, -c)$. Thus we may assume $c \geq 0$. Since the problem is symmetric with respect to a and b, we may assume $a \geq b$. It follows from $ab \geq c^2$ that $a \geq c$, that $c = 0$ if $b = 0$ and that $a + b - 2c \geq 2\sqrt{ab} - 2c \geq 0$ by the Arithmetic-Geometric-Mean Inequality.

We prove the statement by induction on $a + b$. It is trivially true if $a + b = 0$. Assume that it is true when $a + b \leq m$. Consider (a,b,c) where $a + b = m + 1$.

If $c \leq b$, let n = a+b-c. The vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ may be chosen as follows. Let $x_i = 1$ for $1 \leq i \leq a$ and $x_i = 0$ otherwise. Let $y_i = 0$ for $c + 1 \leq b$ and $y_i = 1$ otherwise.

Suppose $c > b$, which implies that $a > c$. Consider $(a+b-2c, b, c-b)$. We have $(a+b-2c)b-(c-b)^2 = ab+bc-c^2 \geq 0$. Moreover, $a+b-2c+b < a+b = m+1$.

By the induction hypothesis, a solution (x, y) exists for $(a + b - 2c, b, c - b)$. It is easy to verify that $(x + y, y)$ is a solution for (a, b, c).

3. Let n be an integer, $n \geq 3$. Let $a_1, a_2, \ldots, a_n$ be real numbers, where $2 \leq a_i \leq 3$ for $i = 1, 2, \ldots, n$. If $s = a_1 + a_2 + \cdots + a_n$, prove that

$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \cdots + \frac{a_n^2 + a_1^2 - a_2^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

Solution

Write

$$A_i \equiv \frac{a_i^2 + a_{i+1}^2 - a_{i+2}^2}{a_i + a_{i+1} - a_{i+2}} = a_i + a_{i+1} + a_{i+2} - \frac{2a_i a_{i+1}}{a_i + a_{i+1} - a_{i+2}}.$$

Since $(a_i - 2)(a_{i+1} - 2) \geq 0$, $-2a_i a_{i+1} \leq -4(a_i + a_{i+1} - 2)$ and

$$A_i \leq a_i + a_{i+1} + a_{i+2} - 4\left(1 + \frac{a_{i+2} - 2}{a_i + a_{i+1} - a_{i+2}}\right).$$

Since $1 = 2 + 2 - 3 \leq a_i + a_{i+1} - a_{i+2} \leq 3 + 3 - 2 = 4$,

$$A_i \leq a_i + a_{i+1} + a_{i+2} - 4\left(1 + \frac{a_{i+2} - 2}{4}\right) = a_i + a_{i+1} - 2.$$

Hence

$$\sum_{i=1}^{n} A_i \leq 2s - 2n.$$

4. Let a, b and c be given positive real numbers. Determine all positive real numbers x, y and z such that

$$x + y + z = a + b + c$$

and

$$4xyz - (a^2x + b^2y + c^2z) = abc.$$

Solution

The second equation is equivalent to

$$4 = \frac{a^2}{yz} + \frac{b^2}{zx} + \frac{c^2}{xy} + \frac{abc}{xyz}.$$

Let $x_1 = a/\sqrt{yz}$, $y_1 = b/\sqrt{zx}$ and $z_1 = c/\sqrt{xy}$. Then $4 = x_1^2 + y_1^2 + z_1^2 + x_1y_1z_1$, where $0 < x_1 < 2, 0 < y_1 < 2, 0 < z_1 < 2$. Regarding the new equation as a quadratic in z_1, the discriminant $(4-x_1^2)(4-y_1^2)$ suggests that we let $x_1 = 2\sin u,\ 0 < u < \pi/2$, and $y_1 = 2\sin v,\ 0 < v < \pi/2$. Now

$$4 = 4\sin^2 u + 4\sin^2 v + z_1^2 + 4\sin u \cdot \sin v \cdot z_1.$$

Hence $(z_1 + 2\sin u \cdot \sin v)^2 = 4(1-\sin^2 u)(1-\sin^2 v)$. In other words

$$|z_1 + 2\sin u \cdot \sin v| = |2\cos u \cdot \cos v|.$$

Since $z_1, \sin u$ and $\sin v$ are all positive, we can discard the absolute values, so

$$z_1 = 2(\cos u \cdot \cos v - \sin u \cdot \sin v) = 2\cos(u+v).$$

Thus

$$2\sin u \cdot \sqrt{yz} = a, \quad 2\sin v \cdot \sqrt{zx} = b,$$
$$2(\cos u \cdot \cos v - \sin u \cdot \sin v)\sqrt{xy} = c.$$

From $x + y + z = a + b + c$,

$$(\sqrt{x}\cos v - \sqrt{y}\cos u)^2 + (\sqrt{x}\sin v + \sqrt{y}\sin u - \sqrt{z})^2 = 0$$

which implies

$$\sqrt{z} = \sqrt{x}\sin v + \sqrt{y}\sin u = \sqrt{x}\frac{y_1}{2} + \sqrt{y}\frac{x_1}{2}.$$

Therefore, $\sqrt{z} = \sqrt{x} \cdot \frac{b}{2\sqrt{zx}} + \sqrt{y} \cdot \frac{a}{2\sqrt{yz}}$, so $z = \frac{a+b}{2}$.

Similarly, $y = \frac{c+a}{2}$ and $x = \frac{b+c}{2}$.

Clearly the triple $(x, y, z) = \left(\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}\right)$ satisfies the given system of equations. Thus it is the unique solution.

5. Let $\mathbb{R}$ be the set of real numbers. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ which simultaneously satisfies the following three conditions?

   (a) There is a positive number M such that $-M \le f(x) \le M$ for all x.

   (b) The value of $f(1)$ is 1.

   (c) If $x \neq 0$, then

$$f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2.$$

Solution

An f that satisfies the three conditions simultaneously does not exist.

Suppose to the contrary that $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditions. Let c be the smallest integral multiple of $\frac{1}{4}$ greater than any $f(x)$. We have $c \ge 2$ since

$$f(2) = f(1 + \frac{1}{1^2}) = f(1) + [f(1)]^2 = 2.$$

Moreover, there exists some x such that $f(x) \geq c - \frac{1}{4}$. Then

$$c \geq f\left(x + \frac{1}{x^2}\right) = f(x) + \left[f\left(\frac{1}{x}\right)\right]^2 \geq c - \frac{1}{4} + \left[f\left(\frac{1}{x}\right)\right]^2.$$

Hence $\left[f\left(\frac{1}{x}\right)\right]^2 \leq \frac{1}{4}$ so that $f\left(\frac{1}{x}\right) \geq -\frac{1}{2}$. Also,

$$c \geq f\left(\frac{1}{x} + x^2\right) = f\left(\frac{1}{x}\right) + [f(x)]^2 \geq -\frac{1}{2} + \left(c - \frac{1}{4}\right)^2.$$

It follows that

$$\frac{1}{2} > \frac{1}{2} - \frac{1}{16} \geq c\left(c - 1 - \frac{1}{2}\right) \geq 2\left(\frac{1}{2}\right) = 1.$$

This is a contradiction.

6. Let n be an integer, $n \geq 3$. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i < x_{i+1}$ for $1 \leq i \leq n-1$. Prove that

$$\frac{n(n-1)}{2} \sum_{i<j} x_i x_j > \left(\sum_{i=1}^{n-1} (n-i) x_i\right) \left(\sum_{j=2}^{n} (j-1) x_j\right).$$

Solution

Let

$$y_i = \sum_{j=i+1}^{n} x_j, \quad y = \sum_{j=2}^{n} (j-1) x_j, \quad c = \frac{n(n-1)}{2},$$

$$z_i = c y_i - (n-i) y.$$

Then

$$\frac{n(n-1)}{2} \sum_{i<j} x_i x_j - \left(\sum_{i=1}^{n-1} (n-i) x_i\right) \left(\sum_{j=2}^{n} (j-1) x_j\right)$$

$$= c \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j - \sum_{i=1}^{n-1} (n-i) x_i y = \sum_{i=1}^{n-1} x_i z_i.$$

It remains to show $\sum_{i=1}^{n-1} x_i z_i > 0$. Since

$$\sum_{i=1}^{n-1} y_i = (x_2 + \cdots + x_n) + (x_3 + \cdots + x_n) + \cdots + x_n = \sum_{j=2}^{n} (j-1) x_j = y$$

and $\sum_{i=1}^{n-1}(n-i) = \frac{n(n-1)}{2} = c$, we have $\sum_{i=1}^{n-1} z_i = 0$. It follows that some z_i are negative. Note that $y = \sum_{j=2}^{n}(j-1)x_j < \sum_{j=2}^{n}(j-1)x_n = cx_n$. Thus $z_{n-1} = cy_{n-1} - y = cx_n - y > 0$. Since

$$\begin{aligned}
\frac{z_{i+1}}{c(n-i-1)} - \frac{z_i}{c(n-i)} &= \frac{y_{i+1}}{n-i-1} - \frac{y_i}{n-i} \\
&= \frac{x_{i+2} + \cdots + x_n}{n-i-1} - \frac{x_{i+1} + \cdots + x_n}{n-i} \\
&> 0,
\end{aligned}$$

we have $\frac{z_1}{n-1} < \frac{z_2}{n-2} < \frac{z_3}{n-3} < \cdots < \frac{z_{n-2}}{2} < z_{n-1}$. Thus there is an integer k such that $z_i \le 0$ for $1 \le i \le k$ and $z_i > 0$ for $k+1 \le i < n$. Then $(x_i - x_k)z_i \ge 0$ for each $1 \le i \le n-1$. Moreover, $(x_i - x_k)z_i > 0$ for some i. Therefore

$$\sum_{i=1}^{n-1} x_i z_i > x_k \sum_{i=1}^{n-1} z_i = 0,$$

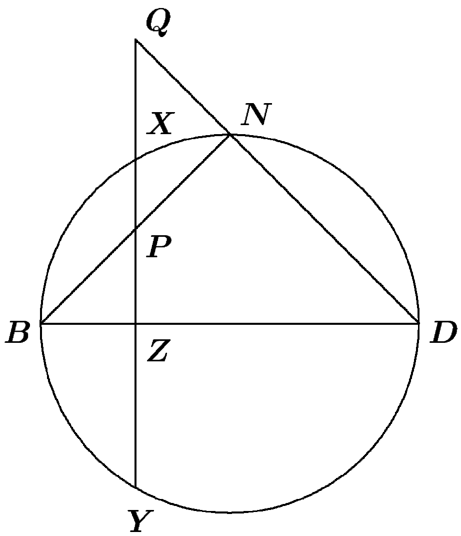
as required.

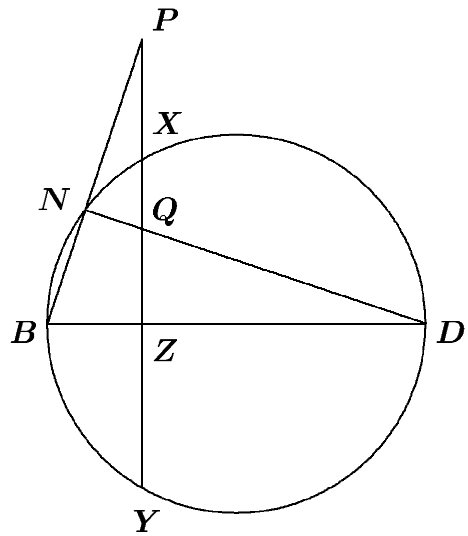
3.2 GEOMETRY

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN, and XY are concurrent.

Solution

Let the lines XY and DN meet at the point Q. Then triangles BPZ and QDZ are similar. Hence $\frac{ZQ}{ZD} = \frac{ZB}{ZP}$ or $ZQ = \frac{ZD \cdot ZB}{ZP}$. Since Z, D, B and P are fixed points, so is Q. By symmetry, AM also passes through Q.





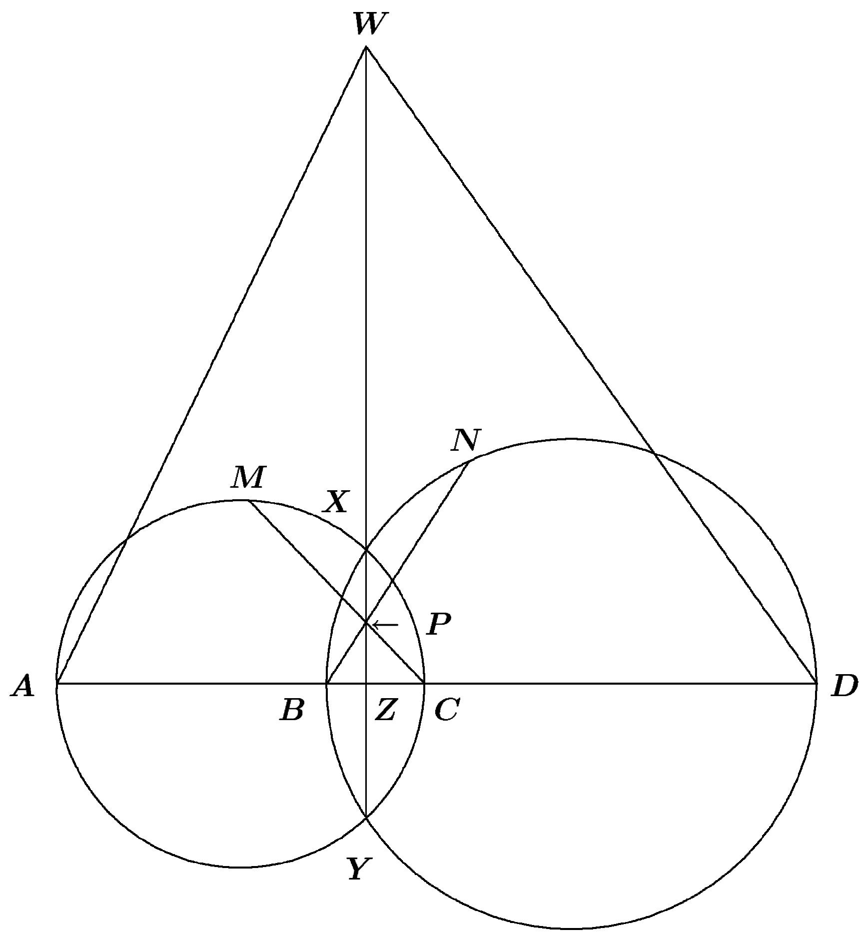
Alternative Solution

Draw a line through A parallel to BP, cutting the line XY at the point W. Then triangles BPZ and AWZ are similar. Hence $\frac{ZP}{ZW} = \frac{ZB}{ZA}$. Since

$$ZA \cdot ZC = ZX \cdot ZY = ZB \cdot ZD,$$

we have $\frac{ZP}{ZW} = \frac{ZC}{ZD}$. It follows that triangles CPZ and DWZ are also similar, so that DW is parallel to CP. Since AM is perpendicular to CP, it is also perpendicular to DW. Similarly, DN is perpendicular to AW. Finally, XY is perpendicular to AD. Hence AM, DN and XY are concurrent at the orthocentre of triangle WAD.

Remark: This is Problem 1 of the 36th IMO on July 19, 1995. It is very easy, and has many different solutions. They may be grouped into two types. The first solution, due independently to Sam Maltby of our Committee and Johannes Notenboom, the leader of the team from the Netherlands, represents those in which only half of the diagram is considered. The second solution, due to Nazar Agakhanov, the leader of the team from Russia, represents those in which the whole diagram is considered. Other techniques used include Menelaus' Theorem and analytic geometry.



2. Let A, B and C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$.

Solution

Let $\triangle A'B'C'$ be such that A, B and C are the respective midpoints of $B'C'$, $C'A'$ and $A'B'$. From the condition on $\triangle XAB$ and $\triangle XAC$, we have $BX^2 - CX^2 = AC^2 - AB^2$. Since B, C and $AC^2 - AB^2$ are fixed, the locus of the point X is a line perpendicular to BC. Moreover, it passes through A' since $A'B^2 - A'C^2 = AC^2 - AB^2$. Similarly, X lies on the line through B' perpendicular to CA, and also on the line through C' perpendicular to AB. It follows that there is a unique position for the point X, namely, the orthocentre of $\triangle A'B'C'$.

3. The incircle of ABC touches BC, CA and AB at D, E and F respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and

touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

Solution

If EF is parallel to BC, then $AB = AC$ and AD is an axis of symmetry of $EFZY$. Hence, the quadrilateral is cyclic. If EF is not parallel to BC, we may assume that the extensions of BC and EF meet at P. By Menelaus' Theorem,

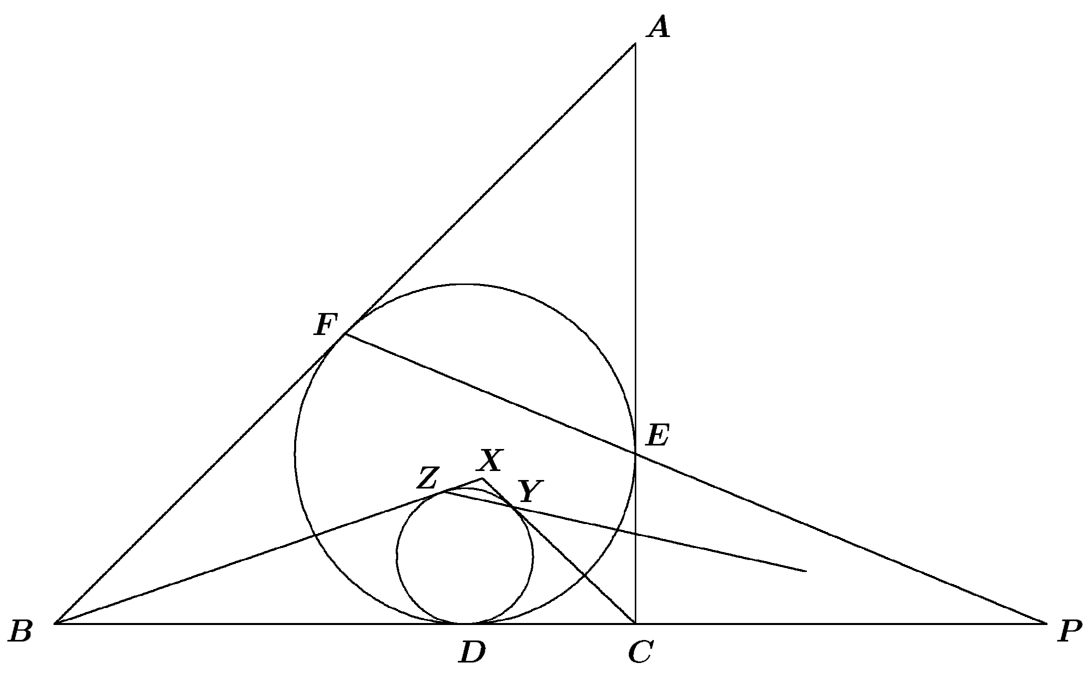
$$\frac{\overrightarrow{AF}}{\overrightarrow{FB}} \cdot \frac{\overrightarrow{BP}}{\overrightarrow{PC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = -1.$$

Since $BZ = BD = BF$, $CY = CD = CE$ and $\frac{AF}{EA} = 1 = \frac{XZ}{YX}$,

$$\frac{\overrightarrow{XZ}}{\overrightarrow{ZB}} \cdot \frac{\overrightarrow{BP}}{\overrightarrow{PC}} \cdot \frac{\overrightarrow{CY}}{\overrightarrow{YX}} = -1.$$

By the converse of Menelaus' Theorem, Z, Y and P are collinear. It follows that $PE \cdot PF = PD^2 = PY \cdot PZ$. Hence $EFZY$ is a cyclic quadrilateral.

Remark: This problem was discarded by the Jury since it has already appeared in a Russian problem book.



4. An acute triangle ABC is given. Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A) and C_1 and C_2 on the side AB (with C_2 between C_1 and B) so that

$$\angle AA_1A_2 = \angle AA_2A_1$$

$$= \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1.$$

The lines AA_1, BB_1 and CC_1 bound a triangle, and the lines AA_2, BB_2 and CC_2 bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

Solution

Let the two triangles be UVW and XYZ as shown in the diagram. Note that since $\angle AB_2X = \angle AC_1U$, $\triangle AB_2B$ and $\triangle AC_1C$ are similar. Hence $\frac{AC_1}{AC} = \frac{AB_2}{AB}$ and $\angle ABB_2 = \angle ACC_1$. Similarly, $\angle BAA_1 = \angle BCC_2$. Now

$$\begin{aligned} \angle A_1VB &= \angle BAA_1 + \angle B_1BB_2 + \angle ABB_2 \\ &= \angle BCC_2 + \angle C_2CC_1 + \angle ACC_1 = \angle ACB. \end{aligned}$$

Similarly, $\angle ACB = \angle AXB_2$ and $\angle A_2ZC = \angle ABC = \angle AUC_1$. By the Sine Formula,

$$\begin{aligned} \frac{AV}{\sin ABV} &= \frac{AB}{\sin A_1VB} = \frac{AB}{\sin ACB} \\ &= \frac{AC}{\sin ABC} = \frac{AC}{\sin A_2ZC} \\ &= \frac{AZ}{\sin ACZ}. \end{aligned}$$

It follows that $AV = AZ$. Similarly, $BW = BX$ and $CU = CY$. Also,

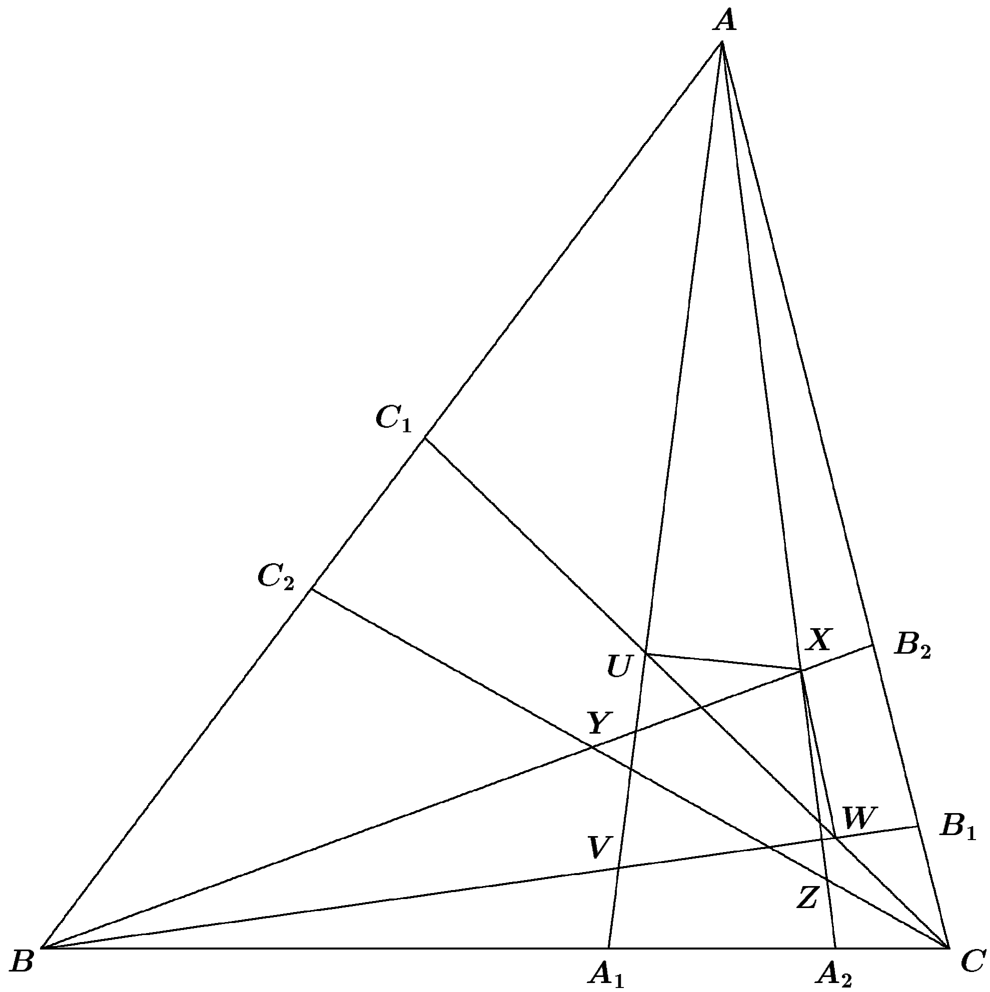
$$\begin{aligned} \frac{AU}{\sin AC_1U} &= \frac{AC_1}{\sin AVC_1} = \frac{AC_1}{AC} \cdot \frac{AC}{\sin ABC} \\ &= \frac{AB_2}{AB} \cdot \frac{AB}{\sin ACB} = \frac{AB_2}{\sin AXB_2} \\ &= \frac{AX}{\sin AB_2X}. \end{aligned}$$

It follows that $AU = AX$. Similarly, $BV = BY$ and $CW = CZ$. In particular, UX is parallel to BC and WX to CA. Consider the quadrilateral $UVWX$. We have

$$\angle AUX = \angle AA_1A_2 = \angle BB_1B_2 = \angle BWX.$$

Hence X lies on the circumcircle of triangle UVW. Similarly, so do Y and Z. It follows that U, V, W, X, Y and Z are concyclic.

Remark: This problem was discarded by the Jury since it has already appeared in a Russian problem book.



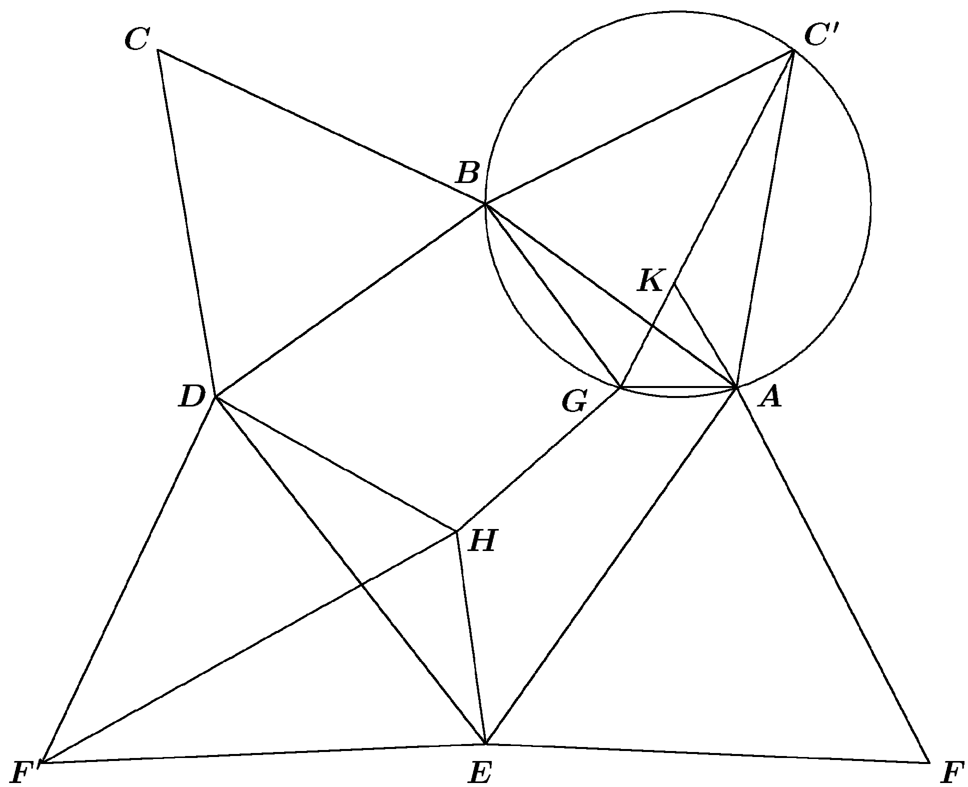
5. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = 60°$. Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Prove that $AG + GB + GH + DH + HE \geq CF$.

Solution

Note that BCD and EFA are equilateral triangles. It follows that BE is an axis of symmetry of $ABDE$. Reflect BCD and EFA about BE to $BC'A$ and $EF'D$ respectively. Since $\angle BGA = 180° - \angle AC'B$, G lies on the circumcircle of ABC'. Hence $\angle AGC' = \angle ABC' = 60°$. Let K be the point on GC' such that KAG is equilateral. Then $\angle C'AK = 60° - \angle BAK = \angle BAG$. Since $C'A = BA$ and $AK = AG$, triangles $C'AK$ and BAG are congruent. It follows that $GC' = GK + KC' = GA + GB$. Similarly, $DH + HE = HF'$. Hence

$$CF = C'F' \leq C'G + GH + HF' = AG + GB + GH + DH + HE,$$

with equality if and only if C', G, H and F' are collinear in that order.



Alternative Solution

The result holds without the condition that $\angle AGB = \angle DHE = 120°$. Let C' and F' be as in the first solution. By Ptolemy's Inequality, $GC' \cdot AB \geq GA \cdot BC' + GB \cdot AC'$ so that $GC' \geq GA + GB$. Similarly, $HF' \geq HD + HE$, It follows that

$$CF = C'F' \leq C'G + GH + HF' \leq AG + GB + GH + HD + HE.$$

Remark: This is Problem 5 of the 36th IMO on July 20, 1995. The first solution is due to Bill Sands of our Committee, who observes that the lemma $GC' = GA + GB$ is a special case of Ptolemy's Theorem which is featured in a 1973 Putnam Mathematics Competition. The second solution is due to Arthur Baragar, a coordinator at the IMO. He uses Ptolemy's Inequality which is a stronger result than Ptolemy's Theorem.

6. Let $A_1A_2A_3A_4$ be a tetrahedron, G its centroid, and A_1', A_2', A_3' and A_4' the points where the circumsphere of $A_1A_2A_3A_4$ intersects GA_1, GA_2, GA_3 and

GA_4 respectively. Prove that

$$GA_1 \cdot GA_2 \cdot GA_3 \cdot GA_4 \leq GA_1' \cdot GA_2' \cdot GA_3' \cdot GA_4'$$

and

$$\frac{1}{GA_1'} + \frac{1}{GA_2'} + \frac{1}{GA_3'} + \frac{1}{GA_4'} \leq \frac{1}{GA_1} + \frac{1}{GA_2} + \frac{1}{GA_3} + \frac{1}{GA_4}.$$

Solution

All summations here range from $i = 1$ to $i = 4$. Let O be the circumcentre and R be the circumradius of $A_1A_2A_3A_4$. By the Power-of-a-point Theorem, $GA_i \cdot GA_i' = R^2 - OG^2$ for $1 \leq i \leq 4$. Hence the desired inequalities are equivalent to

$$(R^2 - OG^2)^2 \geq GA_1 \cdot GA_2 \cdot GA_3 \cdot GA_4 \tag{1}$$

and

$$(R^2 - OG^2) \sum \frac{1}{GA_i} \geq \sum GA_i. \tag{2}$$

Now (1) follows immediately from

$$4(R^2 - OG^2) = \sum GA_i^2 \tag{3}$$

by the Arithmetic-Geometric-Mean Inequality. To prove (3), let P denote the vector from O to the point P. Then

$$\sum (G - A_i)^2 = \sum {A_i}^2 - \sum G^2 + 2G \cdot \sum (G - A_i). \tag{4}$$

This is equivalent to (3) since the last term of (4) vanishes. By Cauchy's Inequality, $4 \sum GA_i^2 \geq (\sum GA_i)^2$ and $\sum GA_i \sum \frac{1}{GA_i} \geq 16$, so that

$$\frac{1}{4} \sum GA_i^2 \sum \frac{1}{GA_i} \geq \frac{1}{16} (\sum GA_i)^2 \sum \frac{1}{GA_i} \geq \sum GA_i.$$

Hence (2) also follows from (3).

7. O is a point inside a convex quadrilateral $ABCD$ of area S. K, L, M and N are interior points of the sides AB, BC, CD and DA respectively. If $OKBL$ and $OMDN$ are parallelograms, prove that $\sqrt{S} \geq \sqrt{S_1} + \sqrt{S_2}$, where S_1 and S_2 are the areas of $ONAK$ and $OLCM$ respectively.

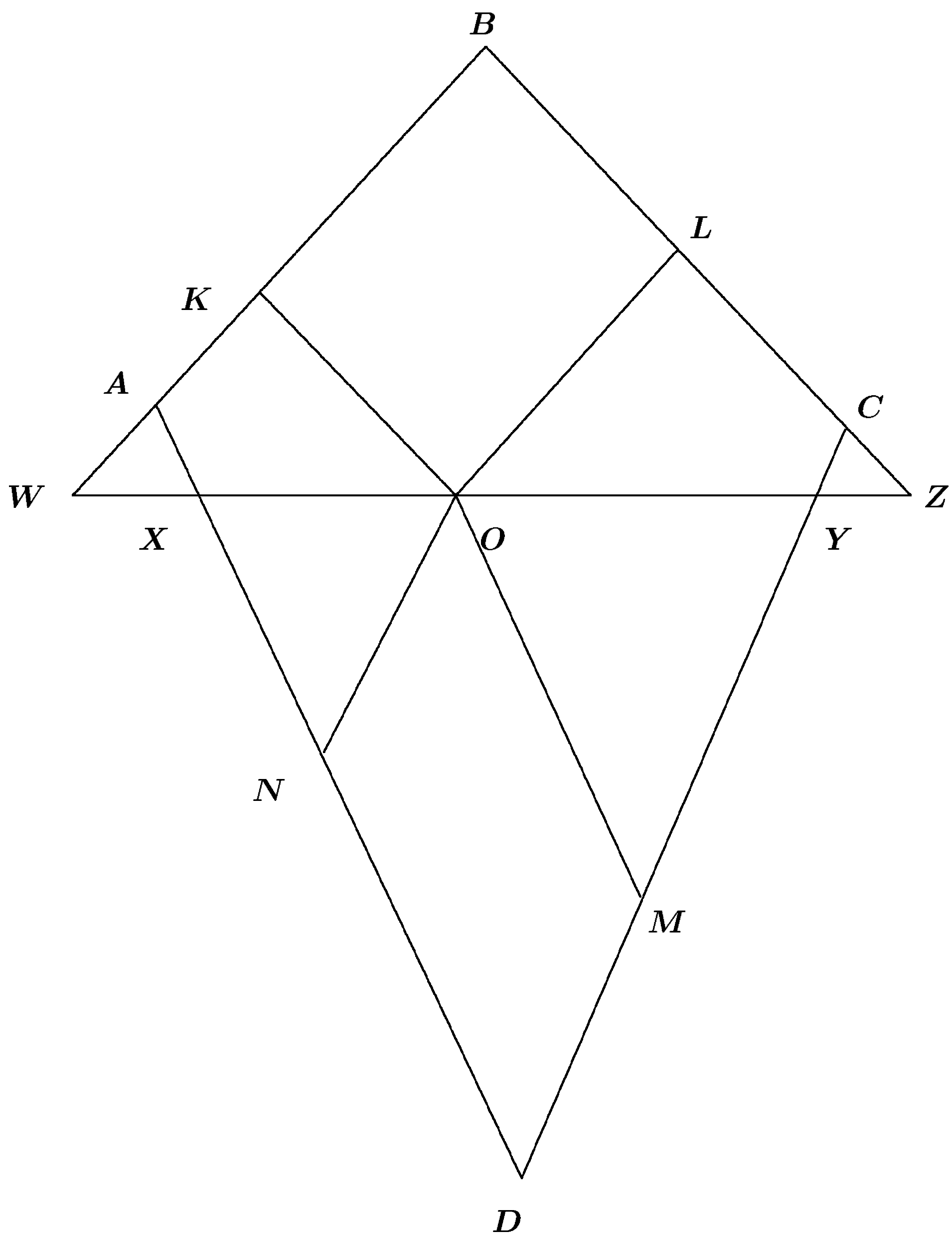
Solution

If O lies on AC, then $ABCD, AKON$ and $OLCM$ are similar, and $AC = AO + OC$. Hence $\sqrt{S} = \sqrt{S_1} + \sqrt{S_2}$. If O does not lie on AC, we may assume that O and D are on the same side of AC. Denote the points of intersection of a line through O with BA, AD, CD and BC by W, X, Y and Z respectively. Initially, let $W = X = A$. Then $\frac{OW}{OX} = 1$ while $\frac{OZ}{OY} > 1$. Rotate the line about O without passing through B, until $Y = Z = C$. Then $\frac{OW}{OX} > 1$ while $\frac{OZ}{OY} = 1$. Hence in some position during the rotation, we have $\frac{OW}{OX} = \frac{OZ}{OY}$. Fix the line there. Let T_1, T_2, P_1, P_2, Q_1 and Q_2 denote the areas of $KBLO, NOMD, WKO, OLZ, ONX$ and YMO respectively. The desired result is equivalent to $T_1 + T_2 \geq 2\sqrt{S_1 S_2}$. Since WBZ, WKO and OLZ are similar, we have

$$\sqrt{P_1} + \sqrt{P_2} = \sqrt{P_1 + T_1 + P_2}\left(\frac{WO}{WZ} + \frac{OZ}{WZ}\right) = \sqrt{P_1 + T_1 + P_2},$$

which is equivalent to $T_1 = 2\sqrt{P_1 P_2}$. Similarly, $T_2 = 2\sqrt{Q_1 Q_2}$.
Since $\frac{OW}{OZ} = \frac{OX}{OY}$, we have $\frac{P_1}{P_2} = \frac{OW^2}{OZ^2} = \frac{OX^2}{OY^2} = \frac{Q_1}{Q_2}$. Denote the common value of $\frac{Q_1}{P_1} = \frac{Q_2}{P_2}$ by k. Then

$$\begin{aligned} T_1 + T_2 = &\ 2\sqrt{P_1 P_2} + 2\sqrt{Q_1 Q_2} = 2\sqrt{P_1 P_2}(1 + k) \\ = &\ 2\sqrt{(1+k)P_1(1+k)P_2} = 2\sqrt{(P_1 + Q_1)(P_2 + Q_2)} \\ \geq &\ 2\sqrt{S_1 S_2}. \end{aligned}$$



8. Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC' and HH' are concurrent, where H and H' are the orthocentres of triangles ABC and $AB'C'$ respectively.

Solution

Since $\angle AB'C' = \angle ABC$, $AB'C'$ and ABC are similar triangles, as are $H'B'C'$ and HBC. Let BB' cut CC' at P. Since

$$\angle BB'C = \angle CC'B, \ \angle PBH = \angle PCH. \tag{1}$$

Since $\angle PB'C' = \angle PCB$, $PB'C'$ and PCB are similar triangles. Complete the parallelogram $BPCD$. Then DBC is congruent to PCB and hence similar to $PB'C'$. It follows that $BHCD$ is similar to $B'H'C'P$, so that BHD is similar to $B'H'P$. Hence

$$\angle HDB = \angle H'PB'. \tag{2}$$

Complete the parallelogram $HPCE$. Then

$$\angle PCH = \angle CHE. \tag{3}$$

Note that $BHED$ is also a parallelogram. Hence

$$\angle DHE = \angle HDB. \tag{4}$$

Now BPH and DCE are congruent triangles. It follows that

$$\angle CDE = \angle PBH \tag{5}$$

and

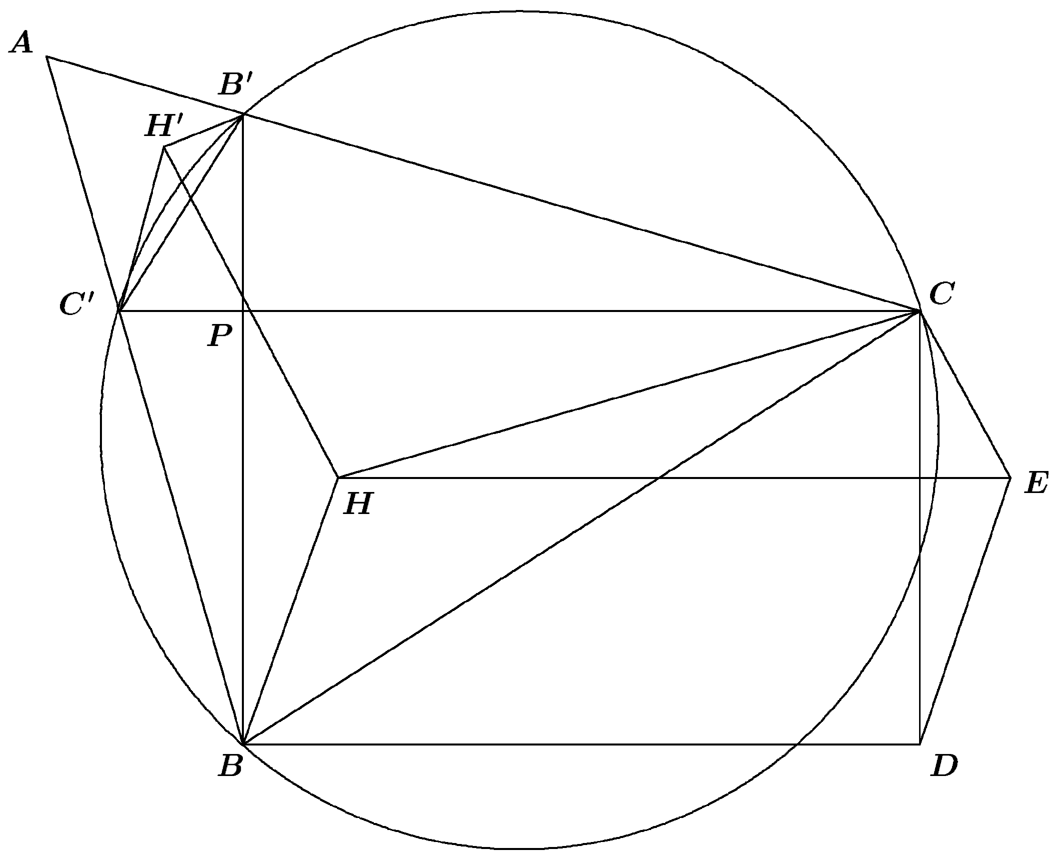
$$\angle BPH = \angle DCE. \tag{6}$$

By (5), (1) and (3), $\angle CDE = \angle CHE$. Hence $HCED$ is cyclic. It now follows that

$$\angle DCE = \angle DHE. \tag{7}$$

By (6), (7), (4) and (2), $\angle BPH = \angle H'PB'$. Hence HH' also passes through P.

Remark: This problem was discarded by the Jury after Nazar Agakhanov, the leader of the team from Russia, claimed that it was used in a Russian contest in 1995.



3.3 NUMBER THEORY & COMBINATORICS

1. Let k be a positive integer. Prove that there are infinitely many perfect squares of the form $n2^k - 7$, where n is a positive integer.

Solution

We first show that, for each k, there exists a positive integer a_k for which $a_k^2 \equiv -7 \pmod{2^k}$. We use induction on k.

Observe that $a_k = 1$ satisfies the condition for $k \leq 3$. If $a_k^2 \equiv -7 \pmod{2^k}$ for some $k > 3$, consider the value of a_k^2 modulo 2^{k+1}. Either

$$a_k^2 \equiv -7 \pmod{2^{k+1}}$$

or

$$a_k^2 \equiv 2^k - 7 \pmod{2^{k+1}}.$$

In the former case, set $a_{k+1} = a_k$. In the latter case, set $a_{k+1} = a_k + 2^{k-1}$. Since $k \geq 3$ and a_k is odd,

$$a_{k+1}^2 = a_k^2 + 2^k a_k + 2^{2k-2} \equiv a_k^2 + 2^k a_k \equiv a_k^2 + 2^k \equiv -7 \pmod{2^{k+1}},$$

by the induction hypothesis.

Finally, we note that the sequence $\{a_k\}$ contains no largest element, since we must have $a_k^2 \geq 2^k - 7$ for each k. Consequently, $\{a_k\}$ contains infinitely many distinct values. This implies the desired result.

2. Let $\mathbb{Z}$ denote the set of all integers. Prove that, for any integers A and B, one can find an integer C for which $M_1 = \{x^2 + Ax + B : x \in \mathbb{Z}\}$ and $M_2 = \{2x^2 + 2x + C : x \in \mathbb{Z}\}$ do not intersect.

Solution

If A is odd, M_1 consists of numbers of the form $x(x + A) + B \equiv B \pmod 2$ while M_2 consists of numbers of the form $2x(x + 1) + C \equiv C \pmod 2$. To ensure that these two sets do not intersect, we can choose $C = B + 1$.

If A is even, M_1 consists of numbers of the form

$$(x + \frac{A}{2})^2 + B - \frac{A^2}{4} \equiv B - \frac{A^2}{4} \quad \text{or} \quad B - \frac{A^2}{4} + 1 \pmod 4,$$

while M_2 consists of numbers of the form $2x(x + 1) + C \equiv C \pmod 4$. Here we can choose $C = B - \frac{A^2}{4} + 2$.

3. Determine all integers $n > 3$ such that there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

   (a) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;

   (b) for each triple $i, j, k (1 \leq i < j < k \leq n)$ the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Solution

We claim that $n = 4$ is the only integer satisfying the conditions of the problem. For $n = 4$, let $A_1A_2A_3A_4$ be a unit square and let $r_1 = r_2 = r_3 = r_4 = 1/6$. It remains to show that no solution exists for $n = 5$, which implies that there are no solutions for any $n \geq 5$.

Suppose to the contrary that there is a solution with $n = 5$. Denote the area of $\triangle A_iA_jA_k$ by $[ijk] = r_i + r_j + r_k, 1 \leq i < j < k \leq 5$. If $A_iA_jA_kA_\ell$ is convex, then $r_i + r_k = r_j + r_\ell$. This follows from $[ijk] + [k\ell i] = [jk\ell] + [\ell ij]$.

We cannot have $r_i = r_j$. If for instance $r_4 = r_5$, then $[124] = [125]$. If A_1 and A_2 are on the same side of A_4A_5, then A_1A_2 must be parallel to A_4A_5. If they are on opposite sides, then A_1A_2 must pass through the midpoint M of A_4A_5. The same can be said about A_2A_3 and A_3A_1. Since A_1, A_2 and A_3 are not collinear, at most one of A_1A_2, A_2A_3 and A_3A_1 can be parallel to A_4A_5, and at most one can pass through M. This is a contradiction.

Consider the convex hull of A_1, A_2, A_3, A_4 and A_5. We have three cases.

First, suppose that the convex hull is a pentagon $A_1A_2A_3A_4A_5$. Since $A_1A_2A_3A_4$ and $A_1A_2A_3A_5$ are convex, our observation yields $r_1 + r_3 = r_2 + r_4$ and $r_1 + r_3 = r_2 + r_5$. Hence $r_4 = r_5$, a contradiction.

Next, suppose that the convex hull is a quadrilateral $A_1A_2A_3A_4$. We may assume that A_5 lies within $A_3A_4A_1$. Then $A_1A_2A_3A_5$ is convex, and we have the same contradiction as before.

Finally, suppose that the convex hull is a triangle $A_1A_2A_3$. Since

$$[124] + [234] + [314] = [125] + [235] + [315],$$

we have $r_4 = r_5$, a contradiction.

Alternative Solution

We proceed as in the first solution up to $r_i + r_k = r_j + r_\ell$ if $A_iA_jA_kA_\ell$ is convex. Assume that $r_1 \geq r_2 \geq r_3 \geq r_4 \geq r_5$. Then [123] is the greatest area among the ten triangles determined by these five points. Hence both A_4 and A_5 lie within the triangle $B_1B_2B_3$ which has A_1, A_2 and A_3 as the midpoints of its sides B_2B_3, B_3B_1 and B_1B_2, respectively.

Suppose both A_4 and A_5 are inside $A_1A_2A_3$. Then

$$[124] + [234] + [314] = [125] + [235] + [315],$$

which implies that $r_4 = r_5$. Since A_4 and A_5 are on the same side of A_1A_2 and of A_2A_3, A_4A_5 must be parallel to both segments. This is a contradiction since A_1, A_2 and A_3 are not collinear.

We may now assume by symmetry that A_4 is in $B_1A_2A_3$. Then $r_1 + r_4 = r_2 + r_3$. If A_5 is also in $B_1A_2A_3$, then $r_1 + r_5 = r_2 + r_3$ so that $r_4 = r_5$. This leads to a contradiction as before. On the other hand, if A_5 is in $A_2A_3B_2B_3$, then $r_4 + r_5 = r_2 + r_3$ so that $r_1 = r_5$. This also leads to a contradiction.

Remark: This is Problem 3 of the 36th IMO on July 19, 1995. The first solution is due to Bill Sands of our Committee. The second solution is due to the proposer of the problem, Karel Horak, leader of the team from the Czech Republic.

4. Find all positive integers x and y such that $x + y^2 + z^3 = xyz$, where z is the greatest common divisor of x and y.

Solution

Let $x = zc$ and $y = zb$, where c and b are relatively prime integers. Then the given Diophantine equation becomes $c + zb^2 + z^2 = z^2cb$. Hence $c = za$ for some integer a, and we have $a + b^2 + z = z^2ab$ or $a = \frac{b^2+z}{z^2b-1}$. If $z = 1$, then $a = \frac{b^2+1}{b-1} = b + 1 + \frac{2}{b-1}$. It follows that $b = 2$ or $b = 3$, so that $(x, y) = (5, 2)$ or $(x, y) = (5, 3)$. If $z = 2$, then $16a = \frac{16b^2+32}{4b-1} = 4b + 1 + \frac{33}{4b-1}$. It follows that $b = 1$ or $b = 3$, so that $(x, y) = (4, 2)$ or $(x, y) = (4, 6)$. In general, $z^2a = \frac{z^2b^2+z^3}{z^2b-1} = b + \frac{b+z^3}{z^2b-1}$. Being a positive integer, $\frac{b+z^3}{z^2b-1} \geq 1$ or $b \leq \frac{z^2-z+1}{z-1}$. If $z \geq 3$, then $\frac{z^2-z+1}{z-1} < z + 1$, so that $b \leq z$. It follows that $a \leq \frac{z^2+z}{z^2-1} < 2$, so that $a = 1$. Now b is an integer solution of $w^2 - z^2w + z + 1 = 0$. This implies that the discriminant $z^4 - 4z - 4$ is a square. However, it lies strictly between $(z^2-1)^2$ and $(z^2)^2$, a contradiction. Hence the only solutions for (x, y) are $(4, 2)$, $(4, 6)$, $(5, 2)$ and $(5, 3)$.

5. At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

Solution

For any two people, let n be the fixed number of others who have exchanged greetings with both. Consider a particular person a. Let B be the set of people who have exchanged greetings with a, and C the set of those who have not. Then there are $3k + 6$ people in B and $9k - 7$ people in C. For any b in B, people who have exchanged greeting with a and b must be in B. Hence b has exchanged greetings with n people in B, and hence with $3k+5-n$ people in C. For any c in C, people who have exchanged greetings with a and c must also be in B. Hence c has exchanged greetings with n people in B. The total number of greeting exchanges between B and C is given by $(3k+6)(3k+5-n) = (9k-7)n$, which simplifies to $9k^2 - 12kn + 33k + n + 30 = 0$. It follows that $n = 3m$ for some positive integer m, and $4m = k + 3 + \frac{9k+43}{12k-1}$. If $k \geq 15$, then $12k - 1 > 9k + 43$ and $4m$ will not be an integer. For $1 \leq k \leq 14$, only $k = 3$ yields an integer value for $\frac{9k+43}{12k-1}$. Hence there can only be 36 people at the party. We now give a construction that such a party can exist.

R	O	Y	G	B	V
V	R	O	Y	G	B
B	V	R	O	Y	G
G	B	V	R	O	Y
Y	G	B	V	R	O
O	Y	G	B	V	R

R = Red
O = Orange
Y = Yellow
G = Green
B = Blue
V = Violet

Let the 36 people sit in a 6 by 6 array, wearing shirts of the colours as indicated in the diagram above. Each person knows only those in the same row, in the same column, or wearing shirts of the same colour. Clearly, each knows exactly 15 others. Let P and Q be any two persons at the party. If they are in the same row, they both know the four other people in that row, the one in P's column and Q's colour, and the one in P's colour and Q's column. The cases where P and Q are in the same column or colour can be verified in an analogous manner. Suppose they are not in the same row, column or colour. Then they both know the six who are respectively in P's row and Q's column, P's row and Q's colour, P's column and Q's row, P's column and Q's colour, P's colour and Q's row, and P's colour and Q's column.

6. Let p be an odd prime. Find the number of subsets A of $\{1, 2, \ldots, 2p\}$ such that

   (a) A has exactly p elements, and

   (b) the sum of all the elements in A is divisible by p.

Solution

For any p-element subset A of $\{1, 2, ..., 2p\}$, denote by $s(A)$ the sum of the elements of A. Of the $\binom{2p}{p}$ such subsets, $B = \{1, 2, ..., p\}$ and $C = \{p+1, p+2, ..., 2p\}$ satisfy $s(B) = s(C) \equiv 0 \pmod p$. For $A \neq B, C$, we have $A \bigcap B \neq \emptyset \neq A \bigcap C$. Partition the $\binom{2p}{p} - 2$ p-element subsets other than B and C into groups of size p as follows. Two subsets A and A' are in the same group if and only if $A' \bigcap C = A \bigcap C$ and $A' \bigcap B$ is a cyclic permutation of $A \bigcap B$ within B. Suppose $A \bigcap B$ has n elements, $0 < n < p$. For some m such that $0 < m < p$,

$A' \bigcap B =$

$$\{x + m : x \in A \bigcap B, x + m \leq p\} \bigcup \{x + m - p : x \in A \bigcap B, x \leq p < x + m\}.$$

Hence $s(A') - s(A) \equiv mn \pmod p$, but mn is not divisible by p. It follows that exactly one subset A in each group satisfies $s(A) \equiv 0 \pmod p$, and the total number of such subsets is

$$p^{-1}\left(\binom{2p}{p} - 2\right) + 2.$$

Alternative Solution

Let ω be a primitive p-th root of unity. Then

$$\prod_{i=1}^{2p}(x - \omega^i) = (x^p - 1)^2 = x^{2p} - 2x^p + 1.$$

Comparing the coefficients of the term x^p, we have

$$2 = \sum \omega^{i_1+i_2+\cdots+i_p} = \sum_{j=0}^{p-1} n_j \omega^j,$$

where the first summation ranges over all subsets $\{i_1, i_2, \ldots, i_p\}$ of $\{1, 2, \ldots, 2p\}$ and n_j in the second summation is the number of such subsets such that $i_1+i_2+\cdots+i_n \equiv j \pmod{p}$. It follows that ω is a root of $G(x) = (n_0-2)+\sum_{j=1}^{p-1} n_j\omega^j$, which is a polynomial of degree $p-1$. Since the minimal polynomial for ω over the field of rational numbers is $F(x) = \sum_{j=0}^{p-1} \omega^j$, which is also of degree $p-1$, $G(x)$ must be a scalar multiple of $F(x)$, so that $n_0 - 2 = n_1 = n_2 = \cdots = n_{p-1}$. Since $\sum_{j=0}^{p} n_j = \binom{2p}{p}$, we have $n_0 = p^{-1}\left(\binom{2p}{p} - 2\right) + 2$.

Remark: This is Problem 6 of the 36th IMO on July 20. The first solution is due to the proposer, Marcin Kuczma, the leader of the team from Poland. The second solution is due to Roberto Dvornicich, the leader of the team from Italy. Nikolay Nikolov, a Bulgarian student, won a special prize for his solution which is essentially along the line of the second one. Nikolay had won two Gold Medals and one Silver Medal at the last three IMO's, and topped off his outstanding career as a competitor by obtaining a perfect score this time.

7. Does there exist an integer $n > 1$ which satisfies the following condition?

   The set of positive integers can be partitioned into n non-empty subsets, such that an arbitrary sum of $n-1$ integers, one taken from each of any $n-1$ of the subsets, lies in the remaining subset.

Solution

Such a number does not exist. Certainly, we cannot have $n = 2$. Suppose $n \geq 3$. Let a and b be distinct numbers in A_1. Let c be any number in A_1 which may be the same as a or b. Suppose $a+c$ and $b+c$ belong to different subsets. If one of them, say $a+c$, also belongs to A_1 while the other belongs to some other subset, say A_3, choose a_i in A_i for $i = 3, 4, \ldots, n$. Then $b+a_3+a_4+\cdots+a_n$ is in A_2, so that $(a+c)+(b+a_3+a_4+\cdots+a_n)+a_4+\cdots+a_n$ is in A_3. On the other hand, $a+a_3+a_4+\cdots+a_n$ is in A_2, so that $(a+a_3+a_4+\cdots+a_n)+(b+c)+a_4+\cdots+a_n$ is in A_1. We have a contradiction. The only other case is where neither $a+c$ nor $b+c$ belongs to A_1, say $a+c$ in A_2 and $b+c$ in A_3. Then $b+(a+c)+a_4+\cdots+a_n$ is in A_3 while $a+(b+c)+a_4+\cdots+a_n$ is in A_2. Again, we have a contradiction. It follows that $a+c$ and $b+c$ must belong to the same subset.

For $i = 1, 2, \ldots, n$, choose x_i in A_i and let $y_i = s - x_i$, where $s = x_1+x_2+\cdots+x_n$. Then y_i is also in A_i. We may assume that s is in A_1. If $x_i = y_i$, then $2x_i = s$ is in A_1. Suppose $x_i \neq y_i$. By what has been proved earlier, $2x_i = x_i + x_i$ is in the same subset as $x_i+y_i = s$, which is in A_1. It follows that A_1 contains all the even numbers. If n is even, then $2+x_3+x_4+\cdots+x_n$ is an even number in A_2, which is a contradiction. Suppose n is odd. Then $x_1+x_3+x_4+\cdots+x_n$ is in A_2 and hence odd. It follows that x_1 must be even, so that A_1 consists of precisely the even numbers. By varying x_1, we can show that all odd integers from a certain point on must belong to A_2 as well as to A_3, which is a contradiction. Hence the condition of the problem cannot be satisfied for any n.

8. Let p be an odd prime. Determine positive integers x and y for which $x \le y$ and $\sqrt{2p}-\sqrt{x}-\sqrt{y}$ is non-negative and as small as possible.

Solution

Let $p = 2n+1$ for some positive integer n. We first prove that $D \equiv \sqrt{2p}-\sqrt{x}-\sqrt{y}$ is non-zero for any positive integers x and y. Otherwise, $2p = x+y+2\sqrt{xy}$. Let b^2 be the largest square which divides x, and c^2 be the largest which divides y. Then $b+c \ge 2$. If $x = ab^2$, then $y = ac^2$. Hence $2p = a(b+c)^2$, but this is a contradiction since $2p$ is not divisible by any square greater than 1. Now

$$D = \frac{2p-(\sqrt{x}+\sqrt{y})^2}{\sqrt{2p}+\sqrt{x}+\sqrt{y}} = \frac{(2p-x-y)^2-4xy}{(\sqrt{2p}+\sqrt{x}+\sqrt{y})(2p-x-y+2\sqrt{xy})}.$$

The numerator is a positive integer. If it is greater than 1, then

$$D \ge \frac{2}{(\sqrt{4n+2}+\sqrt{x}+\sqrt{y})(4n+2-(\sqrt{x}-\sqrt{y})^2)}$$

so that

$$D > \frac{2}{2\sqrt{4n+2}(4n+2)} = \frac{1}{(4n+2)^{3/2}} > \frac{1}{16n^{3/2}}$$

since $4n+2 \le 6n \le 16^{2/3}n$. If the numerator is equal to 1, then $(2p-x-y)^2 = 4xy+1$. Hence $2p-x-y = 2m+1$ for some positive integer m. Let d be the greatest common divisor of m and x and let $m = dh$ and $x = dk$ for relatively prime positive integers h and k. Let g be the greatest common divisor of $m+1$ and y. Then $m+1 = gk$ and $y = gh$. Hence $2p = x+y+m+m+1 = (d+g)(h+k)$. Since p is prime, we must have either $d = g = 1$ or $h = k = 1$. In the former case, $x = k = m+1$ while $y = h = m$, which contradicts $x \le y$. In the latter case, $x = d = m$ and $y = g = m+1$. We have $2p = x+y+2m+1$ so that $p = 2m+1$. It follows that $m = n$. Now

$$D = \frac{1}{\left(\sqrt{4n+2}+\sqrt{n}+\sqrt{n+1}\right)\left(2n+1+2\sqrt{n(n+1)}\right)}$$

so that

$$D < \frac{1}{(\sqrt{4n}+\sqrt{n}+\sqrt{n})(2n+2\sqrt{n\cdot n})} = \frac{1}{16n^{3/2}}.$$

Thus the minimum positive value of D is attained only for $(x,y) = (\frac{p-1}{2}, \frac{p+1}{2})$.

3.4 SEQUENCES

1. Does there exist a sequence $F(1),\ F(2),\ F(3),\ \ldots$ of non-negative integers which simultaneously satisfies the following three conditions?

   (a) Each of the integers $0, 1, 2, \ldots$ occurs in the sequence.

   (b) Each positive integer occurs in the sequence infinitely often.

   (c) For any $n \geq 2$,

$$F(F(n^{163})) \ = \ F(F(n)) + F(F(361)).$$

Solution

Let $F(1) = 0$ and $F(361) = 1$, so that condition (c) becomes:

$$F(F(n^{163})) = F(F(n)) \ \text{ for } \ n \geq 2.$$

For $2 \leq n \leq 360$, let $F(n) = n$. Inductively define $F(n)$ for $n \geq 362$ as follows:

- If $n = m^{163}$ for some m, let $F(n) = F(m)$.
- Otherwise, let $F(n)$ be the smallest number not in $\{F(k) : k < n\}$.

Each non-negative integer appears because there are infinitely many numbers not of the form m^{163}, and each positive integer appears infinitely often since if it appears as $F(n)$ then it also appears as $F(n^{163})$, $F((n^{163})^{163})$, and by an easy induction, as $F(n^{163k})$ for $k \geq 1$.

Condition (c) is satisfied since $F(n) = F(n^{163})$, so

$$F(F(n)) = F(F(n^{163})).$$

2. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

   (a) $x_0 = x_{1995}$;

   (b) $x_{i-1} + \frac{2}{x_{i-1}} \ = \ 2x_i + \frac{1}{x_i}$ for each $i \ = \ 1, \ldots, 1995$.

Solution

The given condition is equivalent to $x_i^2 - \left(\frac{x_{i-1}}{2} + \frac{1}{x_{i-1}}\right) x_i + \frac{1}{2} = 0$, which yields either $x_i \ = \frac{1}{2}x_{i-1}$ or $x_i \ = \ \frac{1}{x_{i-1}}$.

We call the transition from x_{i-1} to x_i a move. Starting from x_0, all possible moves are represented by arrows in Figure 1. The halving moves are represented by solid lines, while the reciprocating moves are represented by broken lines. We have to return to x_0 after exactly 1995 moves.

Note that each row consists of distinct numbers as long as $x_0 \neq 0$, and the two rows of numbers are either disjoint or identical. If they are disjoint, then it is not possible to return to x_0 after an odd number of moves, since each move is between a number enclosed by a circle and one by a square.

It follows that the two rows of numbers are identical. Even in this case, the task is only possible if the numbers in the first row enclosed by circles are identical to those in the second enclosed by squares. Moreover, the numbers in the two rows are descending in opposite direction. It follows that one of the numbers must be equal to 1, so that we can replace Figure 1 by Figure 2.

In order to maximize x_0, there is no point in taking a reciprocal except on the very last move. Of the remaining 1994 moves, exactly half will be made on each side of 1. Hence the maximum value of x_0 is 2^{997}.

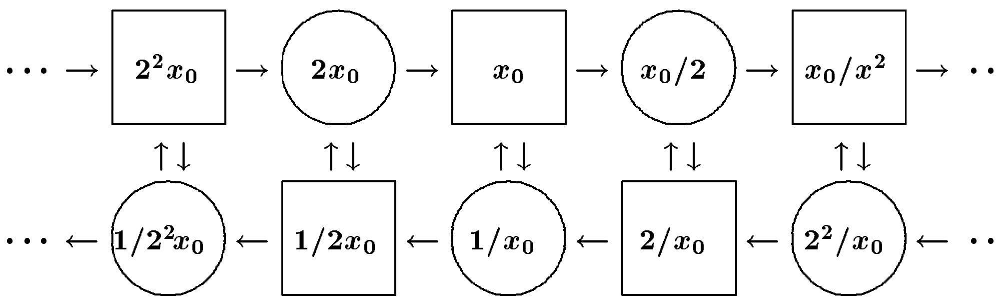


Figure 1.

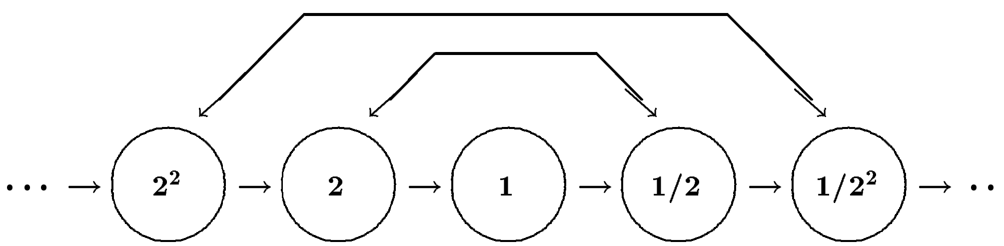


Figure 2.

Alternative Solution

As in the first solution, either $x_i = \frac{x_{i-1}}{2}$ or $x_i = \frac{1}{x_{i-1}}$. For $i \geq 0$, we claim that $x_i = 2^{k_i}x_0^{\epsilon_i}$ for some integer k_i with $|k_i| \leq i$ and $\epsilon_i = (-1)^{k_i+i}$. This is true for $i = 0$, with $k_0 = 0$ and $\epsilon_0 = 1$, and we proceed by induction. If it is true for $i-1$ and $x_i = \frac{1}{2}x_{i-1}$, then we have $k_i = k_{i-1} - 1$ and $\epsilon_i = \epsilon_{i-1}$; while if $x_i = \frac{1}{x_{i-1}}$, then we have $k_i = -k_{i-1}$ and $\epsilon_i = -\epsilon_{i-1}$. In each case, it is immediate that $|k_i| \leq i$ and $\epsilon_i = (-1)^{k_i+i}$. Thus $x_{1995} = 2^k x_0^{\epsilon}$, where $k = k_{1995}$ and $\epsilon = \epsilon_{1995}$, with $0 \leq |k| \leq 1995$ and $\epsilon = (-1)^{1995+k}$. It follows that $x_0 = x_{1995} = 2^k x_0^{\epsilon}$. If k is odd, then $\epsilon = 1$ and we have $2^k = 1$, a contradiction since $k \neq 0$. Thus k must be even, so that $\epsilon = -1$ and $x_0^2 = 2^k$. Since k is even and $|k| \leq 1995$, $k \leq 1994$.

Hence $x_0 \le 2^{997}$. We can have $x_0 = 2^{997}$, $x_i = \frac{1}{2}x_{i-1}$ for $i = 1, 2, \cdots, 1994$, and $x_{1995} = \frac{1}{x_{1994}}$. Then

$$x_{1995} = \frac{1}{2^{-1994}x_0} = x_0$$

as desired.

Remark: This is Problem 4 of the 36th IMO on July 20, 1995. The first solution is due to Johannes Notenboom, the leader of the team from the Netherlands. It is along the line of that of the proposer of the problem, Marcin Kuczma, the leader of the team from Poland, differing only in presentation. The second solution is due to Sam Maltby of our Committee.

3. For an integer $x \ge 1$, let $p(x)$ be the least prime that does not divide x, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$. For x having $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n \ge 0$. Find all n such that $x_n = 1995$.

Solution

It is clear from the definitions of $p(x)$ and $q(x)$ that $q(x)$ divides x for all x, so that

$$x_{n+1} = \frac{x_n}{q(x_n)} \cdot p(x_n)$$

is a positive integer for all n. Also, an easy induction shows that x_n is square-free for all n. Hence we can give each a unique code according to which primes divide it. Let $p_0 = 2, p_1 = 3, p_2 = 5, \ldots$ be the sequence of all primes in increasing order. Let $x > 1$ be any square-free number and let p_m be the largest prime dividing it. Then the code of x is $(1, s_{m-1}, s_{m-2}, \ldots, s_1, s_0)$, where $s_i = 1$ if p_i divides x and $s_i = 0$ otherwise, $0 \le i \le m-1$. Define $f(x) = \frac{xp(x)}{q(x)}$. If the code of x ends in 0, then x is odd, $p(x) = 2$, $q(x) = 1$ and $f(x) = 2x$. The code of $f(x)$ is the same as that of x except that the terminal 0 is replaced by 1. If the code of x ends in $011\ldots1$, then the code of $f(x)$ ends in $100\ldots0$. If we treat the codes as though they are binary numbers, then the code of $f(x)$ can be obtained from that of x by adding 1. From $x_1 = 2$ and $x_{n+1} = f(x_n)$ for $n \ge 2$, the code of x_n is simply the binary representation of the number n. Hence there is a unique n for which $x_n = 1995 = 3 \cdot 5 \cdot 7 \cdot 19$. Since the code of x_n is 10001110 we have $n = 142$ in decimal representation.

4. Suppose that $x_1, x_2, x_3, \ldots$ are positive real numbers for which

$$x_n^n = \sum_{j=0}^{n-1} x_n^j$$

for $n = 1, 2, 3, \ldots$. Prove that for all n,

$$2 - \frac{1}{2^{n-1}} \le x_n < 2 - \frac{1}{2^n}.$$

Solution

For $n = 1$, we have $x_1 = x_1^1 = x_1^0 = 1$, and clearly $2 - 2^0 = 1 < 2 - 2^{-1}$. Suppose now that $n \ge 2$, and let

$$f(x) = x^n - \sum_{j=0}^{n-1} x^j.$$

By Descartes' Rule of Signs, $f(x)$ has a unique positive x_n. For all $n \ge 2$,

$$\begin{aligned}(1 - 2^{-n})^n &\ge (1 - 2^{-n})^{2n-2} = (1 - 2 \cdot 2^{-n} + 2^{-2n})^{n-1} \\ &> (1 - 2^{-(n-1)})^{n-1}.\end{aligned}$$

Thus $(1 - 2^{-n})^n > (1 - 2^{-1})^1 = 1/2$.

Consider $g(x) = (x - 1)f(x) = (x - 2)x^n + 1$. We have

$$g(2 - 2^{-n}) = -2^{-n}(2 - 2^{-n})^n + 1 = -(1 - 2^{-(n+1)})^n + 1 > 0$$

and

$$g(2 - 2^{-(n-1)}) = -2^{-(n-1)}(2 - 2^{-(n-1)})^n + 1 = -2(1 - 2^{-n})^n + 1 < 0.$$

It follows that

$$f(2 - 2^{1-n}) < 0 < f(2 - 2^{-n}).$$

Hence the unique positive root x_n of $f(x)$ satisfies $2 - 2^{1-n} < x_n < 2 - 2^{-n}$, as required.

5. For positive integers n, the numbers $f(n)$ are defined inductively as follows: $f(1) = 1$, and for every positive integer n, $f(n + 1)$ is the greatest integer m such that there is an arithmetic progression of positive integers $a_1 < a_2 < \cdots < a_m = n$ and

$$f(a_1) = f(a_2) = \cdots = f(a_m).$$

Prove that there are positive integers a and b such that $f(an + b) = n + 2$ for every positive integer n.

Solution

By computing the first few values of $f(n)$, we observe the following patterns along with early exceptions:

$$\begin{aligned}
&f(4k) = k, && \text{but } f(8) = 3; && (1)\\
&f(4k+1) = 1, && \text{but } f(5) = f(13) = 2; && (2)\\
&f(4k+2) = k-3, && \text{but } f(1) = 1, f(6) = f(10) = 2, && \\
& && f(14) = f(18) = 3, f(26) = 4; && (3)\\
&f(4k+3) = 2. && && (4)
\end{aligned}$$

We shall prove these statements simultaneously by induction on k.

For $n = 4k$, it is easy to verify that $f(4) = 1$ and $f(8) = 3$. Let $k \geq 3$. Since $f(3) = f(7) = \cdots = f(4k-1) = 2$, we have $f(4k) \geq k$. On the other hand, we have $f(n) \leq \max\{f(m) : m < n\} + 1$. Hence $f(4k) = k$.

For $n = 4k+2$, we have $f(2) = 1$, $f(6) = f(10) = f(22) = 2, f(14) = f(18) = 3$ and $f(26) = 4$. Let $k \geq 7$. Since $f(17) = f(21) = \cdots = f(4k+1) = 1$, we have $f(4k+2) \geq k-3$. On the other hand, if $f(4k+1) = f(4k+1-d) = 1$ and $d > 4$, then $d \geq 8$. Hence $4k+1-d(k-3) \leq 4k+1-8(k-3) = 25-4k < 0$. It follows that $f(4k+2) = k-3$.

For $n = 4k+1$, it is easy to verify that $f(1) = f(9) = 1$ and $f(5) = f(13) = 2$. Let $k \geq 3$. Since $f(4k) = k$ and $f(m) < k$ for all $m < 4k$, $f(4k+1) = 1$.

For $n = 4k+3$, it is easy to verify that $f(3) = f(7) = \cdots = f(31) = 2$. Let $k \geq 8$. Since $f(4k+2) = k-3$ and $f(m) = k-3$ for exactly one $m < 4k+2$, $f(4k+3) = 2$.

It remains to observe that we can take $a = 4$ and $b = 8$ since $f(4n+8) = n+2$ for every positive integer n.

6. Let $\mathbb{N}$ denote the set of all positive integers. Show that there exists a unique function $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(m + f(n)) = n + f(m + 95)$$

for all m and n in $\mathbb{N}$. What is the value of $\sum_{k=1}^{19} f(k)$?

Solution

Let $F(n) = f(n) - 95$ for all $n \geq 1$. Writing k for $m + 95$, the given condition becomes

$$F(k + F(n)) = n + F(k) \qquad (1)$$

for all $n \geq 1$ and $k \geq 96$. Writing m for k in (1) and then adding k to both sides, we have $F(k + n + F(m)) = F(k + F(m + F(n)))$. It follows from (1) that

$$F(k + n) = F(k) + F(n) \qquad (2)$$

for all $n \geq 1$ and $k \geq 96$. We claim that

$$F(96q) = qF(96) \tag{3}$$

for all $q \geq 1$. The case $q = 1$ is trivial, and the inductive step follows immediately from (2). Let m be chosen arbitrarily and let $F(m) = 96q + r,\ 0 \leq r \leq 95$. For any $n \geq 1$, (1), (2) and (3) yield

$$\begin{aligned} m + F(n) &= F(n + F(m)) \;=\; F(n + 96q + r) \\ &= F(n + r) + F(96q) \;=\; F(n + r) + qF(96). \end{aligned} \tag{4}$$

If $1 \leq n \leq 96 - r$, then $1 + r \leq n + r \leq 96$. If $97 - r \leq n \leq 96$ where $r \geq 1$, then $1 \leq n + r - 96 \leq r$. By (2) and (4),

$$\begin{aligned} m + F(n) &= F(n + r - 96 + 96) + qF(96) \\ &= F(n + r - 96) + (q + 1)F(96). \end{aligned} \tag{5}$$

We now sum (4) from $n = 1$ to $n = 96 - r$, and if $r \geq 1$, then we also sum (5) from $n = 97 - r$ to $n = 96$. After cancelling $F(1) + F(2) + \cdots + F(96)$ from both sides, we have

$$96m = F(96)\{q(96 - r) + (q + 1)r\} = F(96)F(m). \tag{6}$$

Setting $m = 96$ in (6), we have $96^2 = [F(96)]^2$. Since $F(96) > 0$, $F(96) = 96$. It now follows from (6) that $F(m) = m$ or $f(m) = m + 95$ for all $m \geq 1$. The desired sum is equal to $1 + 2 + \cdots + 19 + 19 \cdot 95 = 1995$.