33^{rd} International Mathematical Olympiad
Moscow, Russia

Day I
July 15, 1992

1. Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

is a divisor of $abc - 1$.

2. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2$$

for all $x, y \in R$.

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

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4. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set $|A|$. (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

6. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.
   (a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
   (b) Find an integer n such that $S(n) = n^2 - 14$.
   (c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.